- Final: Monday, March $20^{th}$, 3-6 in class
- Grading: on Friday, grades posted on my.ucla by Saturday
- 7 problems covering 5.1-5.5, and 6.1, 6.2, and 6.3 up to thm 6.19.

Recall the Minkowski inequality: $f, g \in L^p \Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$

Replace sums by integrals to obtain:

Thm: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be $\sigma$-finite measure spaces and let $f: X \times Y \to \mathbb{R}_+$ be $\mathcal{A} \otimes \mathcal{B}$ measurable. Then for $1 \leq p < \infty$

$$\left\| \int f(\cdot, y)\, d\nu(y) \right\|_p = \left[ \int \left( \int f(x,y)\, d\nu(y) \right)^p d\mu(x) \right]^{1/p}$$
$$\leq \int \|f(\cdot, y)\|_p\, d\nu(y).$$

Proof: We may assume that $1 < p < +\infty$. ($p=1$ corresponds to Tonelli's theorem). Then take $\underset{=g(x)}{g} \in L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$\int \left( \int f(x,y)\, d\nu(y) \right) |g(x)|\, d\mu(x) = \int \left[ \int f(x,y) |g(x)|\, d\mu(x) \right] d\nu(y)$$
$$\leq \|g\|_q \int \|f(\cdot, y)\|_p\, d\nu(y).$$

Hölder in $X$

By Hölder's converse, we get: $\left\| \int f(\cdot, y)\, d\nu(y) \right\|_p \leq \int \|f(\cdot, y)\|_p\, d\nu(y)$ □

Corollary: If $f(\cdot, y) \in L^p(\mu)$ for a.a. $y$, and $y \longmapsto \|f(\cdot, y)\|_p \in L^1(\mu)$, then $f(x, \cdot) \in L^1(\nu)$ for a.a. $x$ and $x \longmapsto \int f(x,y)\, d\nu(y) \in L^p(\mu)$.

Then Minkowski holds.

Recall: Schur's Lemma: Let $K: X \times Y \to \mathbb{C}$ be measurable and s.t.

"essential supremum"

$$\operatorname{esssup}_y \int |K(x,y)|\, d\mu(x) \leq A < +\infty$$

$$\operatorname{esssup}_x \int |K(x,y)|\, d\nu(y) \leq B < +\infty.$$

Then, $\mathcal{K} f(x) = \int K(x,y) f(y)\, d\nu(y)$ defines a bounded map $L^p \to L^p$, $1 \leq p \leq \infty$ and $\|\mathcal{K}\|_{p \to p} \leq A^{\frac{1}{p}} B^{1 - \frac{1}{p}}$

In particular, if $K(x,y)$ is a convolution kernel: $K(x,y)=f(x-y)$, where (assume $X=Y$, $\mu=\nu$)

$f\in L^1$, then $\mathcal{K}: L^p \to L^p$ with $\|\mathcal{K}\|_{p\to p} \le \|f\|_{L^1}$.

Remark: In $\mathbb{R}$, if $f(x)=\frac{1}{\pi x} \notin L^1$, then nevertheless, one can show that $f*: L^p \to L^p$ for $1<p<+\infty$. (Hilbert Transform)

Example 1: Let $f_n \in L^p(\mathbb{R}^d)$, $1<p<+\infty$, and assume that $f_n(x)\to f(x)$, $n\to\infty$ a.e., and that $\|f_n\|_p \le A < +\infty$, $n=1,2,\dots$ Then, show that $\int f_n g\,dx \to \int fg\,dx$, $\forall g\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$ (ie $f_n\to f$ weakly in $L^p$)

We may assume that $f=0$. When $K$ is large, consider $E_n=\{x: |f_n(x)| > K|g(x)|^{q-1}\}$. Then $\int_{E_n^c} f_n g\,dx \to 0$ since on $E_n^c$,

$|f_n g| \le K|g|^q \in L^1$, by dominated convergence.

Now, $\left|\int_{E_n} f_n g\,dx\right| \le \underbrace{\|f\chi_{E_n}\|_p}_{\le A} \|g\chi_{E_n}\|_q$ and $\int_{E_n} |f_n(x)|^p dx$

$> K^p \int_{E_n} |g(x)|^{(q-1)p} dx = K^p \int_{E_n} |g(x)|^q dx$. Then, we get that

$\|g\chi_{E_n}\|_q \le K^{-\frac{p}{q}} \|f_n\|_p^{p/q} \le \tilde{C} K^{-\frac{p}{q}}$

$\Rightarrow \int_{E_n} f_n g\,dx \le \tilde{C} K^{-\frac{p}{q}}$. Since $K$ can be chosen arbitrarily large, we get the result.

Remark: If, in addition, $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ in $L^p$

Example 2: The distribution function of $f \in L^p(X, \mu)$:

$\lambda(t) = \mu(\{x; |f(x)| > t\}) \downarrow, t > 0.$

If $E = \{(x,t) \in X \times \mathbb{R} : |f(x)|^p > t > 0\}$, then $\iint \chi_E(x,t)\,dx\,dt$

$= \int \Big( \underbrace{\int \chi_E(x,t)\,dt}_{= \int_0^{|f|^p} dt} \Big) dx = \int |f(x)|^p dx = \|f\|_p^p$

On the other hand, $\iint \chi_E(x,t)\,dx\,dt = \int \Big( \underbrace{\int \chi_E(x,t)\,dx}_{\lambda(t^{1/p})} \Big) dt = \int_0^\infty \lambda(t^{1/p})\,dt$

$= \int_0^\infty p\,\lambda(t)\,t^{p-1}\,dt$. We get:

$\int_X |f(x)|^p dx = \int_0^\infty p\,\lambda(t)\,t^{p-1}\,dt, \quad 1 \le p < +\infty$