Econ 203B: Single Equation Models

Unconditional Variance

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The following proposition has already been used a couple times in class and will likely come in handy later.

Proposition 1 $Var\left(\hat{\beta}\right) = E\left[Var\left(\hat{\beta}\middle| X\right)\right] + Var\left[E\left[\hat{\beta}\middle| X\right]\right]$

Proof. First, expanding out the definition of $Var\left(\hat{\beta}\right)$:

$$\begin{aligned}
Var\left(\hat{\beta}\right) &= E\left[\left(\hat{\beta} - E\left[\hat{\beta}\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\right]\right)'\right] \\
&= E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right] + E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right] + E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)'\right] \\
&= E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\right] + E\left[\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)'\right] \\
&\quad + E\left[\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\right] + E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)'\right]
\end{aligned}$$

Noting the following:

$$\begin{aligned}
Var\left(E\left[\hat{\beta}\middle| X\right]\right) &= E\left\{\left(E\left[\hat{\beta}\middle| X\right] - E\left[E\left[\hat{\beta}\middle| X\right]\right]\right)\left(E\left[\hat{\beta}\middle| X\right] - E\left[E\left[\hat{\beta}\middle| X\right]\right]\right)'\right\} \\
&= E\left\{\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)'\right\}
\end{aligned}$$

Where the second equality holds by the law of iterated expectations. Also,

$$\begin{aligned}
E\left[Var\left(\hat{\beta}\middle| X\right)\right] &= E\left\{E\left[\left(\hat{\beta}\middle| X - E\left[\hat{\beta}\middle| X\right]\right)\left(\hat{\beta}\middle| X - E\left[\hat{\beta}\middle| X\right]\right)'\right]\right\} \\
&= E\left\{E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\middle| X\right]\right\} \\
&= E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\right]
\end{aligned}$$

This gives us:

$$\begin{aligned}
Var\left(\hat{\beta}\right) &= Var\left(E\left[\hat{\beta}\middle| X\right]\right) + E\left[Var\left(\hat{\beta}\middle| X\right)\right] + E\left[\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\right] \\
&\quad + E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)'\right]
\end{aligned}$$

It just remains to show that the last two terms are zero:

$$\begin{aligned}
E\left[\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\right] &= E\left[E\left[\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\right]\middle| X\right] \\
&= \left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)E\left[E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)'\right]\middle| X\right] \\
&= \left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)E\left[\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\middle| X\right]\right)'\right] \\
&= \left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)E\left[0\right] = 0
\end{aligned}$$

And finally,

$$\begin{aligned} E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)'\right] &= E\left[E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)'\right]\middle| X\right] \\ &= E\left[E\left[\left(\hat{\beta} - E\left[\hat{\beta}\middle| X\right]\right)\right]\middle| X\right]\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)' \\ &= E\left[\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\middle| X\right]\right)\right]\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)' \\ &= E\left[0\right]\left(E\left[\hat{\beta}\middle| X\right] - E\left[\hat{\beta}\right]\right)' = 0 \end{aligned}$$

Putting this all together, we have:

$$Var\left(\hat{\beta}\right) = Var\left(E\left[\hat{\beta}\middle| X\right]\right) + E\left[Var\left(\hat{\beta}\middle| X\right)\right]$$

Which is the desired result. ■