Econ 203B: Single Equation Models

Solutions for Problem Set 6

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1 Problem 1

a. Let $X \sim N\left(0, \sigma^2\right)$. Derive the following moments of the truncated distribution given that $a < X < b$

$$\begin{array}{l} E\left(X \mid a < X < b\right) \\ Var\left(X \mid a < X < b\right) \end{array}$$

Solution The most obvious first step for this problem is to derive the pdf of $X \mid a < X < b$, which I will now do. First note that

$$\begin{aligned}
F_{X \mid a<X<b}(x) &= \Pr\left(X \leq x \mid a < X < b\right) \\
&= \frac{\Pr\left(\{X \leq x\} \cap \{a < X < b\}\right)}{\Pr\left(a < X < b\right)} \\
&= \begin{cases} \frac{\Pr(a<X<b)}{\Pr(a<X<b)} = 1 & x \geq b \\ \frac{\Pr(a<X<x)}{\Pr(a<X<b)} & a < x < b \\ 0 & x \leq a \end{cases} \\
&= 1_{\{x \geq b\}} + \frac{\Pr\left(a < X < x\right)}{\Pr\left(a < X < b\right)} \cdot 1_{\{a<x<b\}} \\
&= 1_{\{x \geq b\}} + \frac{\int_a^x f(t)\,dt}{\Pr\left(a < X < b\right)} \cdot 1_{\{a<x<b\}}
\end{aligned}$$

Then we have

$$\begin{aligned}
f_{X \mid a<X<b}(x) &= \frac{d}{dx}\left[1_{\{x \geq b\}} + \frac{\int_a^x f_X(t)\,dt}{\Pr\left(a < X < b\right)} \cdot 1_{\{a<x<b\}}\right] \\
&= 1_{\{a<x<b\}} \frac{1}{\Pr\left(a < X < b\right)} \frac{d}{dx} \int_a^x f_X(t)\,dt \\
&= \frac{f_X(x)}{\Pr\left(a < X < b\right)} 1_{\{a<x<b\}} \qquad (1)
\end{aligned}$$

Where I treated the indicator functions as constants under differentiation with respect to x and I used the fundamental theorem of calculus for the second equality. Here, since $X \sim N\left(0, \sigma^2\right)$, we have that

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}.$$

Next, to simplify (1), we note that

$$\begin{aligned}
f_X\left(\frac{x}{\sigma}\right) &= \frac{1}{\sigma}\left[\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right\}\right] \\
&= \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right)
\end{aligned}$$

Where $\phi\left(\frac{x}{\sigma}\right)$ is the pdf for $\frac{X}{\sigma} \sim N(0,1)$. Also, recognizing that since $\frac{X}{\sigma} \sim N(0,1)$,

$$\begin{aligned}
\Pr(a < X < b) &= \Pr\left(\frac{a}{\sigma} < \frac{X}{\sigma} < \frac{b}{\sigma}\right) \\
&= \Pr\left(\frac{X}{\sigma} < \frac{b}{\sigma}\right) - \Pr\left(\frac{X}{\sigma} < \frac{a}{\sigma}\right) \\
&= \Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)
\end{aligned}$$

Where $\Phi(x)$ is the cdf for a standard normal random variable. Plugging this all into (1) gives us the expression:

$$f_{X|a<X<b}(x) = \frac{\frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \cdot 1_{\{a<x<b\}}$$

Now that we have the pdf for $X|\, a < X < b$, we can derive its expected value:

$$\begin{aligned}
E[X|\, a < X < b] &= E\left[X|\, \frac{a}{\sigma} < \frac{X}{\sigma} < \frac{b}{\sigma}\right] \\
&= \int_{-\infty}^{\infty} x f_{X|\frac{a}{\sigma}<\frac{X}{\sigma}<\frac{b}{\sigma}}(x)\, dx \\
&= \int_{-\infty}^{\infty} x \frac{\frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \cdot 1_{\left\{\frac{a}{\sigma}<\frac{x}{\sigma}<\frac{b}{\sigma}\right\}} dx \\
&= \frac{1}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \frac{x}{\sigma}\phi\left(\frac{x}{\sigma}\right) dx \\
&= \frac{\sigma}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \frac{x}{\sigma}\phi\left(\frac{x}{\sigma}\right) d\left(\frac{x}{\sigma}\right) \\
&= \frac{\sigma}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} y\phi(y)\, dy \\
&= \frac{\sigma}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \frac{1}{\sqrt{2\pi}} y \exp\left\{-\frac{y^2}{2}\right\} dy \\
&= \frac{\sigma}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \frac{1}{\sqrt{2\pi}} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} y \exp\left\{-\frac{y^2}{2}\right\} dy
\end{aligned}$$

From here, we note that

$$\frac{d}{dy}\exp\left\{-\frac{y^2}{2}\right\} = -y\exp\left\{-\frac{y^2}{2}\right\}$$

And

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} y \exp\left\{-\frac{y^2}{2}\right\} dy &= -\frac{1}{\sqrt{2\pi}} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \frac{d}{dy}\exp\left\{-\frac{y^2}{2}\right\} dy \\
&= -\frac{1}{\sqrt{2\pi}} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} d\left(\exp\left\{-\frac{y^2}{2}\right\}\right) \\
&= -\frac{1}{\sqrt{2\pi}} \left[\exp\left\{-\frac{y^2}{2}\right\}\right]_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(\frac{a}{\sigma}\right)^2}{2}\right\} - \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(\frac{b}{\sigma}\right)^2}{2}\right\} \\
&= \phi\left(\frac{a}{\sigma}\right) - \phi\left(\frac{b}{\sigma}\right)
\end{aligned}$$

Where I used the Riemann-Stieltjes integral result that

$$\int_a^b df(x) = f(b) - f(a)$$

This gives us:

$$E[X|\, a < X < b] = \sigma \frac{\phi\left(\frac{a}{\sigma}\right) - \phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}$$

In order to calculate this conditional variance, we recognize that:

$$Var[X|\, a < X < b] = E\left[X^2\middle|\, a < X < b\right] - \left(E[X|\, a < X < b]\right)^2 \tag{2}$$

Calculating $E\left[X^2\middle|\, a < X < b\right]$:

$$\begin{aligned}
E\left[X^2\middle|\, a < X < b\right] &= E\left[X^2\middle|\, \frac{a}{\sigma} < \frac{X}{\sigma} < \frac{b}{\sigma}\right] \\
&= \int_{-\infty}^{\infty} x^2 f_{X|\frac{a}{\sigma} < \frac{X}{\sigma} < \frac{b}{\sigma}}(x)\, dx \\
&= \int_{-\infty}^{\infty} x^2 \frac{\frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \cdot 1_{\left\{\frac{a}{\sigma} < \frac{x}{\sigma} < \frac{b}{\sigma}\right\}} dx \\
&= \frac{1}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \frac{x^2}{\sigma} \phi\left(\frac{x}{\sigma}\right) dx \\
&= \frac{\sigma^2}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \left(\frac{x}{\sigma}\right)^2 \phi\left(\frac{x}{\sigma}\right) d\left(\frac{x}{\sigma}\right) \\
&= \frac{\sigma^2}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} y^2 \phi(y)\, dy \\
&= \frac{\sigma^2}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \frac{1}{\sqrt{2\pi}} y^2 \exp\left\{-\frac{y^2}{2}\right\} dy \\
&= \frac{\sigma^2}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} \frac{1}{\sqrt{2\pi}} \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} y^2 \exp\left\{-\frac{y^2}{2}\right\} dy
\end{aligned}$$

And we have, letting

$$\begin{aligned}
u &= y,\ du = dy, \\
dv &= y \exp\left\{-\frac{y^2}{2}\right\} dy = -\frac{d}{dy} \exp\left\{-\frac{y^2}{2}\right\} dy = -d\left(\exp\left\{-\frac{y^2}{2}\right\}\right) \\
v &= -\exp\left\{-\frac{y^2}{2}\right\}
\end{aligned}$$

By integration by parts $\left(\int udv = uv - \int vdu\right)$

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} y^2 \exp\left\{-\frac{y^2}{2}\right\}dy &= \frac{1}{\sqrt{2\pi}}\left[-y\exp\left\{-\frac{y^2}{2}\right\}\right]_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} + \frac{1}{\sqrt{2\pi}}\int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \exp\left\{-\frac{y^2}{2}\right\}dy \\
&= \frac{1}{\sqrt{2\pi}}\left[-\frac{b}{\sigma}\exp\left\{-\frac{\left(\frac{b}{\sigma}\right)^2}{2}\right\} + \frac{a}{\sigma}\exp\left\{-\frac{\left(\frac{a}{\sigma}\right)^2}{2}\right\}\right] \\
&\quad +\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right) \\
&= -\frac{b}{\sigma}\phi\left(\frac{b}{\sigma}\right) + \frac{a}{\sigma}\phi\left(\frac{a}{\sigma}\right) + \Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right) \\
&= \frac{a}{\sigma}\phi\left(\frac{a}{\sigma}\right) - \frac{b}{\sigma}\phi\left(\frac{b}{\sigma}\right) + \Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)
\end{aligned}$$

This gives us:

$$\begin{aligned}
E\left[X^2\middle| a < X < b\right] &= \frac{\sigma^2}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}\left[\frac{a}{\sigma}\phi\left(\frac{a}{\sigma}\right) - \frac{b}{\sigma}\phi\left(\frac{b}{\sigma}\right) + \Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)\right] \\
&= \sigma^2 + \sigma^2\left[\frac{\frac{a}{\sigma}\phi\left(\frac{a}{\sigma}\right) - \frac{b}{\sigma}\phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}\right]
\end{aligned}$$

Which, substituting into (2) gives:

$$\begin{aligned}
Var\left[X\middle| a < X < b\right] &= E\left[X^2\middle| a < X < b\right] - \left(E\left[X\middle| a < X < b\right]\right)^2 \\
&= \sigma^2 + \sigma^2\left[\frac{\frac{a}{\sigma}\phi\left(\frac{a}{\sigma}\right) - \frac{b}{\sigma}\phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}\right] \\
&\quad -\left[\sigma\frac{\phi\left(\frac{a}{\sigma}\right) - \phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}\right]^2 \\
&= \sigma^2\left[1 + \frac{\frac{a}{\sigma}\phi\left(\frac{a}{\sigma}\right) - \frac{b}{\sigma}\phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)} - \left(\frac{\phi\left(\frac{a}{\sigma}\right) - \phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}\right)^2\right]
\end{aligned}$$

b. Consider the following censored regression model:

$$Y_i = \begin{cases} Y_i^* & \text{if } a < Y_i^* < b \\ 0 & \text{otherwise} \end{cases}$$

where the latent variable is given by a linear regression model

$$Y_i^* = X_i\beta_0 + \varepsilon_i$$

where

$$\varepsilon_i | X_i \sim N\left(0, \sigma_0^2\right)$$

Compute $E\left(Y_i\middle| X_i, a < Y_i^* < b\right)$ and $Var\left(Y_i\middle| X_i, a < Y_i^* < b\right)$. Describe 3 methods of estimating β_0 and σ_0^2.

Solution First, recognize that

$$\begin{aligned}
E\left[Y_i \middle| X_i, a < Y_i^* < b\right] &= E\left[0 \middle| X_i, a < Y_i^* < b\right] \cdot \Pr\left(Y_i = 0 \middle| X_i, a < Y_i^* < b\right) \\
&\quad + E\left[Y_i^* \middle| X_i, a < Y_i^* < b\right] \cdot \Pr\left(Y_i = Y_i^* \middle| X_i, a < Y_i^* < b\right) \\
&= E\left[0 \middle| X_i, a < Y_i^* < b\right] \cdot \underbrace{\Pr\left(\left\{Y_i^* \le a\right\} \cup \left\{Y_i^* \ge b\right\} \middle| a < Y_i^* < b\right)}_{=0} \\
&\quad + E\left[Y_i^* \middle| X_i, a < Y_i^* < b\right] \cdot \underbrace{\Pr\left(a < Y_i^* < b \middle| a < Y_i^* < b\right)}_{=1} \\
&= E\left[Y_i^* \middle| X_i, a < Y_i^* < b\right]
\end{aligned}$$

Next, we know that $\varepsilon_i | X_i \sim N\left(0, \sigma_0^2\right)$ and $\varepsilon_i | X_i = Y_i^* - X_i\beta_0 | X_i \sim N\left(0, \sigma_0^2\right)$. Let $Z_i \equiv Y_i^* - X_i\beta_0$. Then we have that

$$\begin{aligned}
E\left[Y_i^* \middle| X_i, a < Y_i^* < b\right] &= E\left[X_i\beta_0 + Z_i \middle| X_i, a - X_i\beta_0 < Z_i < b - X_i\beta_0\right] \\
&= X_i\beta_0 + E\left[Z_i \middle| X_i, a - X_i\beta_0 < Z_i < b - X_i\beta_0\right] \\
&= X_i\beta_0 + E\left[Z_i \middle| X_i, a^* < Z_i < b^*\right]
\end{aligned}$$

Recall from part (a) the result that

$$E\left[X \middle| a < X < b\right] = \sigma \frac{\phi\left(\frac{a}{\sigma}\right) - \phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}$$

Where $X \sim N\left(0, \sigma^2\right)$. Here, we recognize that $Z \sim N\left(0, \sigma_0^2\right)$ and get the result that

$$\begin{aligned}
E\left[Z_i \middle| X_i, a^* < Z_i < b^*\right] &= \sigma_0 \frac{\phi\left(\frac{a^*}{\sigma_0}\right) - \phi\left(\frac{b^*}{\sigma_0}\right)}{\Phi\left(\frac{b^*}{\sigma_0}\right) - \Phi\left(\frac{a^*}{\sigma_0}\right)} \\
&= \sigma_0 \frac{\phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right) - \phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right)}
\end{aligned}$$

This gives us:

$$\begin{aligned}
E\left[Y_i \middle| X_i, a < Y_i^* < b\right] &= E\left[Y_i^* \middle| X_i, a < Y_i^* < b\right] \\
&= X_i\beta_0 + \sigma_0 \frac{\phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right) - \phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right)}
\end{aligned}$$

Next, we have:

$$\begin{aligned}
Var\left(Y_i \middle| X_i, a < Y_i^* < b\right) &= Var\left(Y_i^* \middle| X_i, a < Y_i^* < b\right) \\
&= Var\left(X_i\beta_0 + Z_i \middle| X_i, a < X_i\beta_0 + Z_i < b\right) \\
&= Var\left(Z_i \middle| X_i, a - X_i\beta_0 < Z_i < b - X_i\beta_0\right) \\
&= Var\left(Z_i \middle| X_i, a^* < Z_i < b^*\right) \\
&= E\left[Z_i^2 \middle| X_i, a^* < Z_i < b^*\right] - \left(E\left[Z_i \middle| X_i, a^* < Z_i < b^*\right]\right)^2
\end{aligned}$$

Where I used, in the third equality, the fact that $Var\left(a + Y\right) = Var\left(Y\right)$ when a is a constant and Y is a random variable. Recall the result from part (a) that

$$E\left[X^2 \middle| a < X < b\right] = \sigma^2 + \sigma^2 \left[\frac{\frac{a}{\sigma}\phi\left(\frac{a}{\sigma}\right) - \frac{b}{\sigma}\phi\left(\frac{b}{\sigma}\right)}{\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)}\right]$$

Substituting in Z_i^2 for X^2 and, of course, a^* for a and b^* for b, we get:

$$\begin{aligned}
E\left[\left. Z_i^2 \right| X_i, a^* < Z_i < b^*\right] &= \sigma_0^2 + \sigma_0^2 \left[\frac{\frac{a^*}{\sigma_0}\phi\left(\frac{a^*}{\sigma_0}\right) - \frac{b^*}{\sigma_0}\phi\left(\frac{b^*}{\sigma_0}\right)}{\Phi\left(\frac{b^*}{\sigma_0}\right) - \Phi\left(\frac{a^*}{\sigma_0}\right)}\right] \\
&= \sigma_0^2 + \sigma_0^2 \left[\frac{\frac{a-X_i\beta_0}{\sigma_0}\phi\left(\frac{a-X_i\beta_0}{\sigma_0}\right) - \frac{b-X_i\beta_0}{\sigma_0}\phi\left(\frac{b-X_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{b-X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a-X_i\beta_0}{\sigma_0}\right)}\right]
\end{aligned}$$

Putting this all together, we get:

$$\begin{aligned}
Var\left(\left. Y_i \right| X_i, a < Y_i^* < b\right) &= Var\left(\left. Y_i^* \right| X_i, a < Y_i^* < b\right) \\
&= \sigma_0^2 + \sigma_0^2 \left[\frac{\frac{a-X_i\beta_0}{\sigma_0}\phi\left(\frac{a-X_i\beta_0}{\sigma_0}\right) - \frac{b-X_i\beta_0}{\sigma_0}\phi\left(\frac{b-X_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{b-X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a-X_i\beta_0}{\sigma_0}\right)}\right] \\
&\quad - \left\{X_i\beta_0 + \sigma_0 \frac{\phi\left(\frac{a-X_i\beta_0}{\sigma_0}\right) - \phi\left(\frac{b-X_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{b-X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a-X_i\beta_0}{\sigma_0}\right)}\right\}^2
\end{aligned}$$

The last part of the question asks us to consider three estimation procedures for β_0 and σ_0^2 in this model.

MLE In order to use the ML approach, we need first to set up the likelihood function. Here, we recognize that since

$$Y_i$$

$$\begin{aligned}
L(\beta, \sigma) &= \prod_{i=1}^{n} f_{Y_i}(y_i) \\
&= \prod_{i: y_i \le a \cup y_i \ge b} \Pr\left(Y_i^* \le a \cup Y_i^* \ge b\right) \prod_{i: a < y_i < b} f_{Y_i | a < Y_i^* < b}(y_i) \Pr\left(a < Y_i^* < b\right) \\
&= \prod_{i: y_i \le a \cup y_i \ge b} \Pr\left(Y_i^* \le a \cup Y_i^* \ge b\right) \prod_{i: a < y_i < b} \frac{f_{Y_i}(y_i)}{\Pr\left(a < Y_i^* < b\right)} \Pr\left(a < Y_i^* < b\right) \\
&= \prod_{i: y_i \le a \cup y_i \ge b} \Pr\left(Y_i^* \le a \cup Y_i^* \ge b\right) \prod_{i: a < y_i < b} f_{Y_i}(y_i) \\
&= \prod_{i: y_i \le a \cup y_i \ge b} \left[1 - \left(\Phi\left(\frac{b - x_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - x_i\beta_0}{\sigma_0}\right)\right)\right] \cdot \prod_{i: a < y_i < b} \frac{1}{\sigma_0}\phi\left(\frac{y_i - x_i\beta_0}{\sigma_0}\right)
\end{aligned}$$

Where I used in the third equality the result from part (a) that

$$f_{X | a < X < b}(x) = \frac{f_X(x)}{\Pr(a < X < b)} 1_{\{a < x < b\}}$$

And the fact that for all $a < y_i < b$,

$$f_{Y_i^* | a < Y_i^* < b}(y_i) = f_{Y_i | a < Y_i^* < b}(y_i).$$

The maximum likelihood estimators $\hat{\beta}^{MLE}$ and $\hat{\sigma}^2$ are thus the solutions to the first order conditions of the problem:

$$\max_{\beta, \sigma^2} \prod_{i: y_i \le a \cup y_i \ge b} \left[1 - \left(\Phi\left(\frac{b - x_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - x_i\beta_0}{\sigma_0}\right)\right)\right] \cdot \prod_{i: a < y_i < b} \frac{1}{\sigma_0}\phi\left(\frac{y_i - x_i\beta_0}{\sigma_0}\right)$$

Alternatively, I could have defined a dummy variable

$$D_i = \begin{cases} 1 & y_i \leq a \cup y_i \geq b \\ 0 & a < y_i < b \end{cases}$$

And defined the likelihood function as

$$L\left(\beta, \sigma^2\right) = \prod_{i=1}^{n} \left[1 - \left(\Phi\left(\frac{b - x_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - x_i\beta_0}{\sigma_0}\right)\right)\right]^{D_i} \left[\frac{1}{\sigma_0}\phi\left(\frac{y_i - x_i\beta_0}{\sigma_0}\right)\right]^{1-D_i}$$

And taken first order conditions to obtain the MLEs.

NLS A general nonlinear least squares estimator is the solution to

$$\min_{\beta,\sigma^2} \sum_{i=1}^{n} \left(Y_i - E\left[Y_i | X_i\right]\right)^2$$

Here, we need only derive the expression $E\left[Y_i | X_i\right]$.

$$\begin{aligned} E\left[Y_i | X_i\right] &= E\left[Y_i | X_i, a < Y_i^* < b\right] \Pr\left(a < Y_i^* < b\right) \\ &\quad + \underbrace{E\left[0 | X_i, Y_i^* \leq a \cup Y_i^* \geq b\right]}_{=0} \Pr\left(Y_i^* \leq a \cup Y_i^* \geq b\right) \\ &= E\left[Y_i | X_i, a < Y_i^* < b\right] \Pr\left(a < Y_i^* < b\right) \\ &= \left[X_i\beta_0 + \sigma_0 \frac{\phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right) - \phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right)}{\Phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right)}\right] \left[\Phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right)\right] \\ &= \sigma_0 \left[\phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right) - \phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right)\right] + X_i\beta_0 \left[\Phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right)\right] \end{aligned}$$

Since this model exhibits conditional heteroskedasticity, we could use the more efficient estimator that solves

$$\min_{\beta,\sigma^2} \sum_{i=1}^{n} \frac{\left(Y_i - E\left[Y_i | X_i\right]\right)^2}{Var\left(Y_i | X_i\right)}$$

Heckman's Two-Step Method Finally, we have Heckman's two-step method. In this setting, which is more general than that covered in class, I believe we would need to

1. Use MLE to obtain the estimates $\hat{\gamma}^{MLE}$ and $\hat{\delta}^{MLE}$ for the expressions $\frac{1}{\sigma_0} \equiv \gamma_0$ and $\frac{\beta_0}{\sigma_0} \equiv \delta_0$.

2. For the model

$$\begin{aligned} Y_i &= E\left[Y_i | X_i\right] + \varepsilon_i \\ &= \sigma_0 \left[\phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right) - \phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right)\right] \\ &\quad + X_i\beta_0 \left[\Phi\left(\frac{b - X_i\beta_0}{\sigma_0}\right) - \Phi\left(\frac{a - X_i\beta_0}{\sigma_0}\right)\right] + \varepsilon_i \end{aligned}$$

Substitute in the results from step 1 and use OLS to estimate the parameters $\hat{\sigma}$ and $\hat{\beta}$ for

$$Y_i = \sigma_0 \left[\phi\left(a\hat{\gamma} - X_i\hat{\delta}\right) - \phi\left(b\hat{\gamma} - X_i\hat{\delta}\right)\right] + X_i\beta_0 \left[\Phi\left(b\hat{\gamma} - X_i\hat{\delta}\right) - \Phi\left(a\hat{\gamma} - X_i\hat{\delta}\right)\right] + \varepsilon_i$$