Econ 203B: Single Equation Models

Solutions to Midterm 2006

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Question 1

Ture or False. Prove your answer. Whenever necessary, provide the assumptions you are making in proving your claims.

(1) In the classical linear regression model, the OLS estimator is efficient within the class of all linear unbiased estimators.

Solution: This claim is true by the Gauss-Markov theorem. Let $\hat{\beta} = (X'X)^{-1} X'Y \equiv AY$ be the OLS estimator of the classical linear regression model. Let $\tilde{\beta} = \tilde{A}Y$ be any other linear estimator of β satisfying $E\left[\tilde{\beta}\middle| X\right] = \beta$. Then, we have

$$\begin{aligned}
\beta &= E\left[\hat{\beta}\middle| X\right] = E\left[AY| X\right] = AE\left[Y| X\right] = AX\beta \Rightarrow AX = I \\
\beta &= E\left[\tilde{\beta}\middle| X\right] = E\left[\tilde{A}Y\middle| X\right] = \tilde{A}E\left[Y| X\right] = \tilde{A}X\beta \Rightarrow \tilde{A}X = I
\end{aligned}$$

Define $D = \tilde{A} - A$ or $\tilde{A} = D + A$. Then we have

$$I = \tilde{A}X = (D + A)X = DX + AX = DX + I$$

Or

$$\begin{aligned}
DX &= 0 \\
DX(X'X)^{-1} &= 0 \\
DA' &= 0
\end{aligned}$$

And thus

$$AD' = 0$$

This gives us

$$\begin{aligned}
\tilde{A}\tilde{A}' &= (D + A)(D + A)' \\
&= DD' + AD' + DA' + AA' \\
&= DD' + AA'
\end{aligned}$$

Next, note that

$$\begin{aligned}
Var\left(\hat{\beta}\middle| X\right) &= Var\left(AY| X\right) = AVar\left(Y| X\right)A' = \sigma^2 AA' \\
Var\left(\tilde{\beta}\middle| X\right) &= Var\left(\tilde{A}Y\middle| X\right) = \tilde{A}Var\left(Y| X\right)\tilde{A}' = \sigma^2 \tilde{A}\tilde{A}'
\end{aligned}$$

And thus,

$$\begin{aligned}
Var\left(\tilde{\beta}\middle| X\right) &= Var\left(\hat{\beta}\middle| X\right) + \sigma^2 DD' \\
Var\left(\tilde{\beta}\middle| X\right) - Var\left(\hat{\beta}\middle| X\right) &= \sigma^2 DD'
\end{aligned}$$

Which is positive semi-definite. Therefore, $Var\left(\hat{\beta}\middle| X\right) \leq Var\left(\tilde{\beta}\middle| X\right)$ in matrix sense, or $\hat{\beta}$ is the best linear unbiased estimator.

(2) In the simple regression model, the coefficient of determination is nothing but the squared sample correlation coefficient between the dependent and the independent variable.

Solution: This statement is true. Let $Y_i = \alpha + \beta X_i + \varepsilon_i$ be a simple regression model. The sample correlation coefficient between Y_i and X_i is given by

$$r = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2}}$$

And thus

$$r^2 = \frac{\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right]^2}{\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right]\left[\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right]}$$

The coefficient of determination (centered R^2) is

$$R^2 = \frac{\hat{Y}'M^0\hat{Y}}{Y'M^0Y} = \frac{\sum_{i=1}^{n}\left(\hat{Y}_i - \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i\right)^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2}$$

Since $\hat{Y}_i = \hat{\alpha} + \hat{\beta}X_i$, where $\hat{\beta} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}$, we have

$$\begin{aligned}
R^2 &= \frac{\sum_{i=1}^{n}\left(\hat{\alpha} + \hat{\beta}X_i - \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\alpha} + \hat{\beta}X_i\right)\right)^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2} \\
&= \frac{\sum_{i=1}^{n}\left(\hat{\alpha} + \hat{\beta}X_i - \hat{\alpha} - \hat{\beta}\bar{X}\right)^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2} = \hat{\beta}^2\frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2} \\
&= \frac{\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right]^2}{\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right]^2}\frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2} \\
&= \frac{\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)\right]^2}{\left[\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right]\left[\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right]} = r^2
\end{aligned}$$

(3) The GLS estimator is biased and inefficient when in fact the assumptions of the classical linear regression model hold.

Solution: This statement is false. Suppose the assumptions of the classical linear regression model hold:

$$\begin{aligned}
1.\ \ E\left[Y|\,X\right] &= X\beta \\
2.\ \ E\left[\varepsilon\varepsilon'|\,X\right] &= \sigma^2 I_n \equiv V
\end{aligned}$$

$$3.\ \ rank\left(X'X\right) = k$$

The GLS estimator is given by

$$\begin{aligned}
\hat{\beta}^{GLS} &= \left(X'V^{-1}X\right)^{-1}X'V^{-1}Y \\
&= \left(X'\frac{1}{\sigma^2}I_nX\right)^{-1}X'\frac{1}{\sigma^2}I_nY \\
&= \sigma^2\left(X'X\right)^{-1}\frac{1}{\sigma^2}X'Y \\
&= \left(X'X\right)^{-1}X'Y = \hat{\beta}^{OLS}
\end{aligned}$$

And we know that when all the assumptions of the classical linear regression model hold, $\hat{\beta}^{OLS}$ is both unbiased and efficient. Thus, $\hat{\beta}^{GLS}$ is both unbiased and efficient.

(4) The OLS estimator is biased when the independent variables in a regression are subject to classical measurement error.

NOTE: For part (4) consider the simple regression model:

$$y_i^* = \alpha + \beta x_i^* + \varepsilon_i$$

where instead of observing x_i^*, we observe x_i given by

$$x_i = x_i^* + v_i$$

where $E[\varepsilon_i] = E[v_i] = E[\varepsilon_i | x_i^*] = E[v_i | x_i^*] = 0$ and ε_i and v_i are independent of each other. Assume i.i.d. sampling across i.

Solution: This statement is true.

$$\begin{aligned} y_i^* &= \alpha + \beta(x_i - v_i) + \varepsilon_i \\ &= \alpha + \beta x_i + \underbrace{\varepsilon_i - \beta v_i}_{u_i} \\ &= \alpha + \beta x_i + u_i \\ &\equiv X_i \gamma + u_i \end{aligned}$$

Where $X_i = \begin{bmatrix} 1 & x_i \end{bmatrix}$ and $\gamma = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. OLS will give us

$$\begin{aligned} \hat{\gamma} &= (X'X)^{-1} X'Y^* \\ &= (X'X)^{-1} X'(X\gamma + \varepsilon - \beta v) \\ &= \gamma + (X'X)^{-1} X'\varepsilon - (X'X)^{-1} X'\beta v \end{aligned}$$

Taking expectations,

$$E[\hat{\gamma} | X] = \gamma + (X'X)^{-1} X'E[\varepsilon | X] - (X'X)^{-1} X'\beta E[v | X] \neq 0$$

Unless $E[\varepsilon | X] = 0$ and $E[v | X] = 0$, but $E[v | X] = 0$ implies $E[x_i v_i] = 0$, but we can see that

$$E[x_i v_i] = E[(x_i^* + v_i) v_i] = E[x_i^* v_i] + E[v_i^2] = E[v_i^2] = \sigma_v^2 \neq 0$$

Thus, unless $\sigma_v^2 = 0$ (i.e. there really is no measurement error), then $E[\hat{\gamma} | X] \neq \gamma$.

Question 2

Derive the maximum likelihood estimator of σ^2 in the classical normal linear regression model

$$Y | X \sim N(X\beta, \sigma^2 I_n)$$

Is it unbiased? What is its variance? Is it efficient? Prove your claims.

Solution: Since $Y | X \sim N(X\beta, \sigma^2 I_n)$, it follows that $Y_i | X \sim N(X_i\beta, \sigma^2)$ are independent. (If two random variables are uncorrelated and jointly normal, then they are independent) Thus,

$$\begin{aligned} L(\beta, \sigma^2) &= \prod_{i=1}^{n} f_{Y_i|X_i}(Y_i) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(Y_i - X_i\beta)^2\right\} \\ &= \frac{1}{(\sigma^2)^{\frac{n}{2}} (2\pi)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - X_i\beta)^2\right\} \end{aligned}$$

Taking logs

$$\log L\left(\beta,\sigma^2\right) = -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(Y_i - X_i\beta\right)^2$$

The first order conditions yield

$$\left(\sigma^2\right): -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\left(\hat{\sigma}^2\right)^2}\sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)^2 = 0$$

Or

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}$$

Is $\hat{\sigma}^2$ unbiased?

$$\begin{aligned}
E\left[\hat{\sigma}^2\middle| X\right] &= \frac{1}{n}E\left[\hat{\varepsilon}'\hat{\varepsilon}\middle| X\right] = \frac{1}{n}E\left[\varepsilon' M_X\varepsilon\middle| X\right] = \frac{1}{n}E\left[tr\left(\varepsilon' M_X\varepsilon\right)\middle| X\right] \\
&= \frac{1}{n}E\left[tr\left(M_X\varepsilon\varepsilon'\right)\middle| X\right] = \frac{1}{n}tr\left(E\left[M_X\varepsilon\varepsilon'\middle| X\right]\right) = \frac{1}{n}tr\left(M_X E\left[\varepsilon\varepsilon'\middle| X\right]\right) \\
&= \frac{1}{n}tr\left(M_X\sigma^2 I_n\right) = \frac{\sigma^2}{n}tr\left(M_X\right) = \sigma^2\frac{n-k}{n} \neq \sigma^2
\end{aligned}$$

Where I used the fact that

$$\begin{aligned}
tr\left(M_X\right) &= tr\left(I_n - X\left(X'X\right)^{-1}X'\right) = n - tr\left(X\left(X'X\right)^{-1}X'\right) \\
&= n - tr\left(\left(X'X\right)^{-1}X'X\right) = n - tr\left(I_k\right) = n-k
\end{aligned}$$

Since $E\left[\hat{\sigma}^2\middle| X\right] = \sigma^2\frac{n-k}{n}$, it follows by the law of iterated expectations that $E\left[\hat{\sigma}\right] = \sigma^2\frac{n-k}{n} \neq \sigma^2$. Therefore, $\hat{\sigma}^2$ is not unbiased.
What is the variance of $\hat{\sigma}^2$? Recall that $\frac{\hat{\varepsilon}'\hat{\varepsilon}}{\sigma^2} \sim \chi^2(n-k)$. Therefore,

$$\begin{aligned}
Var\left(\frac{\hat{\varepsilon}'\hat{\varepsilon}}{\sigma^2}\right) &= 2(n-k) \\
\frac{1}{\left(\sigma^2\right)^2}Var\left(\hat{\varepsilon}'\hat{\varepsilon}\right) &= 2(n-k) \\
\frac{n^2}{\left(\sigma^2\right)^2}Var\left(\frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}\right) &= 2(n-k) \\
Var\left(\hat{\sigma}^2\right) &= \frac{2(n-k)\sigma^4}{n^2}
\end{aligned}$$

Since this estimator is not unbiased, the notion of efficiency is not well-defined, so we cannot say anything about its efficiency.