Econ 203B: Single Equation Models

Solutions for 2000 Midterm

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1. (16 points) A four-variable regression using quarterly data from 1958 to 1976 inclusive gave an estimated equation:

$$\hat{Y} = 2.20 + 0.104X_2 - 3.48X_3 + 0.34X_4$$

The explained sum of squares was 109.6 and the error sum of squares 18.48. When the equation was re-estimated with three seasonal dummies added to the specification, the explained sum of squares rose to 114.8.

a. (8 points) Test for the presence of seasonality.

Solution Here, we are trying to test the hypothesis $H_0 : \gamma_1 = \gamma_2 = \gamma_3 = 0$, where γ_i is the seasonal dummy for season i. Recall that $F_0 = \frac{\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}}{\hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}}\frac{n-k}{p}$. Here, since we have quarterly data from 1958 to 1976, it follows that $n = 76$. With the additional three seasonal dummies, $k = 7$. We are imposing three restrictions, so $p = 3$.

Finally, we need to compute $\hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}$. We know that $\hat{\varepsilon}'_R\hat{\varepsilon}_R = 18.48$ (error sum of squares of the restricted model) and $\hat{Y}'_R M^0 \hat{Y}_R = 109.6$ (explained sum of squares of the restricte dmodel. This gives us

$$\underbrace{Y'M^0Y}_{TSS_R} = \underbrace{\hat{Y}'_R M^0 \hat{Y}_R}_{RSS_R} + \underbrace{\hat{\varepsilon}'_R\hat{\varepsilon}_R}_{ESS_R} = 109.6 + 18.48 = 128.08$$

But noting that $TSS_R = TSS_{UR}$ (that is, the total sum of squares does not change when we change the model), we have:

$$\begin{aligned} \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR} &= Y'M^0Y - \hat{Y}'_{UR}M^0\hat{Y}_{UR} \\ &= 128.08 - 114.8 = 13.28 \end{aligned}$$

Therefore, our F statistic is:

$$\begin{aligned} F_0 &= \frac{\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}}{\hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}}\frac{n-k}{p} \\ &= \frac{18.48 - 13.28}{13.28}\frac{76-7}{3} \\ &= 9.006 \end{aligned}$$

Comparing this to $c^*_{0.05,F(3,69)} = 2.74$, we reject the null that there is no seasonality.

b. (8 points) Two further regressions based on the original specification were run for the subperiods 1958-I to 1968-IV and 1969-I to 1976-IV, yielding error sum of squares 9.32 and 7.46, respectively. Test for the constancy of the relationship over the two subperiods.

Solution This is a standard Chow test problem. Let $D_t = \begin{cases} 1 & 1958 - I \text{ to } 1968 - IV \\ 0 & 1969 - I \text{ to } 1976 - IV \end{cases}$. The unrestricted model is:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + \delta_1 D_t + \delta_2 X_{2t} D_t + \delta_3 X_{3t} D_t + \delta_4 X_{4t} D_t + \varepsilon_t$$

Here, we want to test: $H_0 : \Gamma\beta = \gamma_0$ where

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_4 \\ \delta_1 \\ \vdots \\ \delta_4 \end{bmatrix}, \gamma_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The restricted model here is the original model of the question:

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + \varepsilon_t$$

The F statistic for this test is:

$$F_0 = \frac{\hat{\varepsilon}_R'\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}}{\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}} \frac{n-k}{p}$$

Here, $n = 76$, $k = 8$, $p = 4$, and we know that the restricted model has the same error sum of squares as did the restricted model of part (a) (since it is the same model!): $\hat{\varepsilon}_R'\hat{\varepsilon}_R = 18.48$. Further, we know that $\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR} = 9.32 + 7.46 = 16.78$. This gives us the value for the F statistic:

$$F_0 = \frac{18.48 - 16.78}{16.78} \frac{68}{4} = 1.7223$$

And the critical value is $c^*_{0.05,F(4,68)} = 2.51$. Since $F_0 \leq c^*_{0.05,F(4,68)}$, we fail to reject the null and conclude that there was no change in the fundamental relationship between the two subperiods.

2. (40 points) A regression of real investment (in trillions of dollars) on a constant, a time trend $(1, 2, 3, \ldots)$, real GNP (in trillions of dollars), interest rate (in percentage points) and inflation rate (in percentage points, computed as the percentage change in the CPI - Consumer Price Index) using annual time series data from 1968-1982 procues the following results:

Regression results for an investment equation

Sum of squared residuals 45×10^{-5}

Variable	Coefficient	Standard Error
Constant	-0.5091	0.0551
Time	-0.0166	0.0019
Real GNP	0.6704	0.0550
Interest Rate	-0.0023	0.0012
Inflation Rate	-9×10^{-5}	0.0013

Estimated Covariance Matrix

Constant	Time	Real GNP	Interest	Inflation
0.00304				
0.00014	39×10^{-7}			
-0.00304	-0.0001	0.0030		
56×10^{-7}	-29×10^{-8}	-73×10^{-7}	15×10^{-7}	
-32×10^{-7}	-26×10^{-9}	-23×10^{-7}	-75×10^{-8}	18×10^{-7}

HINT: Pay attention to the units in which each of the variables is expressed.

a. (10 points) Interpret the results of the above regression. Which of the coefficients are statistically significant at the 5% level of significance?

b. (10 points) Are the data consistent with the hypothesis that a \$1 billion increase in real GNP will be associated with an equal increase in investment?

c. (10 points) Compute a forecast for real investment in 1983 assuming forecasts of \$3100 billion for nominal GNP, 212 for the CPI which implies an inflation rate of 2.12%, and 10% for the interest rate.

d. (10 points) Describe in as much detail as possible how you would construct a 95% confidence interval for your forecast of part (c).

3. (30 points) You have the position of political analyst at a local TV station. In view of upcoming congressional elections, you are asked to comment on the results from the 1996 elections. You have data on the percentage of votes received by Democratic candidates among all votes cast for House of Representatives candidates for each one of the 50 states. In addition, for each one of the states you have data on the unemployment rate, and you know whether Bill Clinton appeared there to campaign for congressional candidates.

a. (15 points) You believe that the percentage of votes received by Democratic candidates in each state may be explained by the state's unemployment rate, by whether Bill Clinton campaigned in that state, and by the state's location in the country in four different regions: Northeast, South, Midwest, and West. Write down a regression model for the determination of the percentage of democratic votes per state that expresses these beliefs. Interpret the coefficient estimates of the model which you would obtain if you would actually use your data to estimate the regression model. Describe how you would test for the following hypotheses using the estimated coefficients and their estimated covariance matrix:

Solution Here, we have $n = 50$. Define

$$\begin{aligned} V_i &= \text{\% votes received by Democratic candidates} \\ U_i &= \text{unemployment rate} \\ C_i &= \begin{cases} 1 & \text{if Clinton campaigned in state } i \\ 0 & \text{else} \end{cases} \end{aligned}$$

Also, define the following regional dummy variables:

$$\begin{aligned} NE_i &= \begin{cases} 1 & \text{if state } i \text{ is in the Northeast} \\ 0 & \text{else} \end{cases} \\ MW_i &= \begin{cases} 1 & \text{if state } i \text{ is in the Midwest} \\ 0 & \text{else} \end{cases} \\ S_i &= \begin{cases} 1 & \text{if state } i \text{ is in the South} \\ 0 & \text{else} \end{cases} \\ W_i &= \begin{cases} 1 & \text{if state } i \text{ is in the West} \\ 0 & \text{else} \end{cases} \end{aligned}$$

The model I would estimate is:

$$V_i = \beta_1 U_i + \beta_2 C_i + \beta_3 NE_i + \beta_4 MW_i + \beta_5 S_i + \beta_6 W_i + \varepsilon_i$$

For all that follows, we have that $n = 50$ and $k = 6$.

(i) Clinton's campaigning didn't matter.

Solution This hypothesis is the same as $H_0 : \beta_2 = 0$, which would involve doing a simple $t - test$ on the β_2 coefficient for statistical significance:

$$t_0^{\hat{\beta}_2} = \frac{\hat{\beta}_2}{se\left(\hat{\beta}_2\right)}$$

Where

$$se\left(\hat{\beta}_2\right) = \sqrt{\frac{\hat{\varepsilon}'\hat{\varepsilon}}{44}\Gamma\left(X'X\right)^{-1}\Gamma'},\ \Gamma = \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

And, of course comparing $\left|t_0^{\hat{\beta}_2}\right|$ to $c^*_{0.025,t(44)} = 2.02$. If $\left|t_0^{\hat{\beta}_2}\right| \leq c^*_{0.025,t(44)}$, we would conclude that Clinton's campaigning didn't matter. Otherwise, we would conclude that it be either a good idea (if $\hat{\beta}_2 > 0$) or a bad idea (if $\hat{\beta}_2 < 0$) to convince Clinton to help with the campaigning.

(ii) The entire country voted uniformly: there are no regional differences.

Solution This hypothesis is the same as $H_0 : \beta_3 = \beta_4 = \beta_5 = \beta_6$. In matrix notation, this hypothesis is $H_0 : \Gamma\beta = \gamma_0$ where

$$\Gamma = \left[\begin{array}{cccccc} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right],\ \gamma_0 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

It can be seen that $p \equiv rank\left(\Gamma\right) = 3$. To test this hypothesis, construct the test statistic:

$$F_0 = \frac{1}{p}\left(\Gamma\hat{\beta} - \gamma_0\right)'\left(\hat{\sigma}^2\Gamma\left(X'X\right)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta} - \gamma_0\right)$$

Where $\hat{\sigma}^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-k}$. Comparing this to the critical value $c^*_{0.05,F(3,44)} = 2.82$, one would conclude that if $F_0 \leq c^*_{0.05,F(3,44)}$, then the country voted uniformly. Otherwise, it did not.

(iii) The Northeast and the Midwest (the "frostbelt") voted uniformly.

Solution Here, we have $H_0 : \beta_3 = \beta_4$. There are two possible ways to test this hypothesis: 1) use the t-statistic or 2) use the F-statistic. I will do both. Here, we have $p = 1$ and as in the other problems, $n - k = 44$. This gives us:

$$t_0^{\hat{\beta}_3 - \hat{\beta}_4} = \frac{\hat{\beta}_3 - \hat{\beta}_4}{se\left(\hat{\beta}_3 - \hat{\beta}_4\right)}$$

Where

$$\begin{aligned} se\left(\hat{\beta}_3 - \hat{\beta}_4\right) &= \sqrt{\widehat{Var}\left(\hat{\beta}_3 - \hat{\beta}_4\right)} \\ &= \sqrt{\widehat{Var}\left(\hat{\beta}_3\right) + \widehat{Var}\left(\hat{\beta}_4\right) - 2\widehat{Cov}\left(\hat{\beta}_3, \hat{\beta}_4\right)} \end{aligned}$$

All the terms of which can be read directly off the estimated variance-covariance matrix. Then, comparing $\left|t_0^{\hat{\beta}_3 - \hat{\beta}_4}\right|$ to $c^*_{0.025,t(44)} = 2.02$, we would conclude that the Frostbelt voted uniformly if $\left|t_0^{\hat{\beta}_3 - \hat{\beta}_4}\right| \leq c^*_{0.025,t(44)}$ and conclude the opposite otherwise.

Alternatively, this same hypothesis can be tested using the F statistic:

$$\begin{aligned} F_0 &= \frac{1}{p}\left(\Gamma\hat{\beta} - \gamma_0\right)'\left(\hat{\sigma}^2\Gamma\left(X'X\right)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta} - \gamma_0\right) \\ &= \frac{\left(\hat{\beta}_3 - \hat{\beta}_4\right)^2}{\hat{\sigma}_2\Gamma\left(X'X\right)^{-1}\Gamma'} \end{aligned}$$

And compared it to $c^*_{0.05,F(1,44)} = 4.06$, concluding that the Frostbelt voted uniformly if $F_0 \leq c^*_{0.05,F(1,44)}$ and concluding the opposite otherwise. There are advantages to using either formula, but I prefer to use the F statistic, because it is more general than the t statistic, so it doesn't require memorizing two different formulas.

(iv) The frostbelt voted uniformly, the "sunbelt" (the South and the West) voted uniformly, but the frostbelt and the sunbelt did not necessarily vote uniformly.

Make sure you give all necessary details for carrying on the tests (detailed description of the test statistics, critical values, "rules" for testing)

Solution Here, $H_0 : \beta_3 = \beta_5,\ \beta_4 = \beta_6$ or, in matrix notation, $H_0 : \Gamma\beta = \gamma_0$ where

$$\Gamma = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix},\ \gamma_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For this particular hypothesis, $p = rank\,(\Gamma) = 2$. I would construct the F statistic:

$$F_0 = \left(\Gamma\hat{\beta} - \gamma_0\right)' \left(\hat{\sigma}^2\Gamma\left(X'X\right)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta} - \gamma_0\right)$$

And compare it to the critical value $c^*_{0.05,F(2,44)}$ from which I would conclude that the Sunbelt voted uniformly and the Frostbelt voted uniformly if $F_0 \leq c^*_{0.05,F(2,44)}$ and conclude otherwise if $F_0 > c^*_{0.05,F(2,44)}$.

b. (15 points) In addition you may think that the effect of Clinton's campaign may be different depending on whether the state is in the Northeast, South, Midwest, or West. Write down a regression model for the determination of the percentage of democratic votes per state that also captures this effect. Interpret the coefficient estimates of the model which you would obtain if you would actually use your data to estimate the regression model. Describe how you would test for the hypothesis that Clinton's appearance had the same effect in all regions. Give two formulations for the test statistic you would use to test this hypothesis.

Solution For this part, I would estimate the following model (defining all variables as in part (a)):

$$\begin{aligned} V_i &= \beta_1 U_i + \beta_2 NE_i + \beta_3 MW_i + \beta_4 S_i + \beta_5 W_i \\ &\quad +\beta_6 C_i \cdot NE_i + \beta_7 C_i \cdot MW_i + \beta_8 C_i \cdot S_i + \beta_9 C_i \cdot W_i + \varepsilon_i \end{aligned} \tag{1}$$

To best interpret the coefficients, take the derivative with respect to C_i :

$$\frac{\partial V_i}{\partial C_i} = \beta_6 NE_i + \beta_7 MW_i + \beta_8 S_i + \beta_9 W_i$$

Taking partial derivatives with respect to the dummy variables:

$$\frac{\partial^2 V_i}{\partial C_i \partial NE_i} = \beta_6;\ \frac{\partial^2 V_i}{\partial C_i \partial MW_i} = \beta_7;\ \frac{\partial^2 V_i}{\partial C_i \partial S_i} = \beta_8;\ \frac{\partial^2 V_i}{\partial C_i \partial W_i} = \beta_9$$

The interpretation of this is that β_6 measures the effect of Clinton's appearance on the percentage of people voting Democrat in states in the Northeast. β_7 measures the effect of Clinton's appearance on the percentage of people voting Democrat in states in the Midwest. Similarly, β_8 and β_9 measure the effect of Clinton's appearance on the percentage of people voting Democrat in the South and in the West, respectively.

If we want to test the hypothesis $H_0 : \beta_6 = \beta_7 = \beta_8 = \beta_9$, we can construct the F statistic in multiple ways. In a departure from the solutions to part (a), I will test this in a slightly different manner (which in question 4, we will see is equivalent.) Estimate (1) using the standard methods to obtain $\hat{\varepsilon}_{UR}$. Then, impose the restriction that $\beta_6 = \beta_7 = \beta_8 = \beta_9 \equiv \delta$ and note that $\forall i,\ NE_i + MW_i + S_i + W_i = 1$. This gives us:

$$\begin{aligned} V_i &= \beta_1 U_i + \beta_2 NE_i + \beta_3 MW_i + \beta_4 S_i + \beta_5 W_i \\ &\quad +\delta C_i \left(NE_i + MW_i + S_i + W_i\right) + u_i \\ &= \beta_1 U_i + \beta_2 NE_i + \beta_3 MW_i + \beta_4 S_i + \beta_5 W_i + \delta C_i + u_i \end{aligned} \tag{2}$$

This is what is called the "restricted model." Estimate (2) using the standard methods and obtain $\hat{\varepsilon}_R$. Then, since this involves three restriction, $p = 3$. Also, $n = 50$, and $k = 9$. Construct the test statistic:

$$F_0 = \frac{n-k}{p} \frac{\hat{\varepsilon}_R'\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}}{\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}} = 17\frac{\hat{\varepsilon}_R'\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}}{\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}}$$

And compare F_0 to $c^*_{0.05,F(3,41)}$, concluding that Clinton's speech has the same effect regardless of where it is given if $F_0 \le c^*_{0.05,F(3,41)}$ and conclude that there are regional differences for the effect of Clinton's speech if $F_0 > c^*_{0.05,F(3,41)}$.

4. (14 points)

a. (7 points) Obtain the Restricted Least Squares estimator for the k-variate CR model $Y = X\beta + \varepsilon$ under the restriction that $\Gamma\beta = \gamma_0$ where Γ is $p \times k$ matrix of constants of full row rank.

Solution Here, we want to find:

$$\begin{aligned} \hat{\beta}_R &= \underset{b:\Gamma b = \gamma_0}{\arg\min} \left\{ (Y - Xb)' (Y - Xb) \right\} \\ &= \underset{b:\Gamma b = \gamma_0}{\arg\min} \left\{ \frac{1}{2} (Y - Xb)' (Y - Xb) \right\} \end{aligned}$$

Since this is a constrained optimization problem, we can set this up using the Lagrange multiplier method:

$$\begin{aligned} L(b, X) &= \frac{1}{2} (Y - Xb)' (Y - Xb) + \lambda (\Gamma b - \gamma_0) \\ &= \frac{1}{2} (Y'Y - b'X'Y - Y'Xb + b'X'Xb) + \lambda (\Gamma b - \gamma_0) \\ &= \frac{1}{2} (Y'Y - 2Y'Xb + b'X'Xb) + \lambda (\Gamma b - \gamma_0) \end{aligned}$$

Taking first order conditions

$$(b) \; : \; \frac{1}{2} \left(-2X'Y + 2X'X\hat{\beta}_R \right) + \Gamma'\lambda' = 0 \tag{1}$$

$$(\lambda) \; : \; \Gamma\hat{\beta}_R - \gamma_0 = 0 \tag{2}$$

Solving (1) for $\hat{\beta}_R$:

$$-X'Y + X'X\hat{\beta}_R + \Gamma'\lambda' = 0$$

$$\begin{aligned} X'X\hat{\beta}_R &= X'Y - \Gamma'\lambda' \\ \hat{\beta}_R &= (X'X)^{-1} X'Y - (X'X)^{-1} \Gamma'\lambda' \\ &= \hat{\beta}_{UR} - (X'X)^{-1} \Gamma'\lambda' \end{aligned} \tag{3}$$

Subsituting this into (2) and solving for λ',

$$\Gamma\hat{\beta}_{UR} - \Gamma (X'X)^{-1} \Gamma'\lambda' - \gamma_0 = 0$$

$$\begin{aligned} \Gamma (X'X)^{-1} \Gamma'\lambda' &= \Gamma\hat{\beta}_{UR} - \gamma_0 \\ \lambda' &= \left(\Gamma (X'X)^{-1} \Gamma' \right)^{-1} \left(\Gamma\hat{\beta}_{UR} - \gamma_0 \right) \end{aligned}$$

And substituting this back into (3) gives us the final result:

$$\hat{\beta}_R = \hat{\beta}_{UR} - (X'X)^{-1} \Gamma' \left(\Gamma (X'X)^{-1} \Gamma' \right)^{-1} \left(\Gamma\hat{\beta}_{UR} - \gamma_0 \right)$$

Or

$$\hat{\beta}_{UR} - \hat{\beta}_R = (X'X)^{-1}\Gamma'\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)$$

b. (7 points) Use the estimator of part (a) to obtain the relationship between the F statistic used for testing the hypothesis that $\Gamma\beta = \gamma_0$ and the sum of squared residuals of the restricted and the unrestricted regression.

Solution Let $\hat{\varepsilon}_{UR} = Y - X\hat{\beta}_{UR}$ and $\hat{\varepsilon}_R = Y - X\hat{\beta}_R$. Then, $\hat{\varepsilon}_R - \hat{\varepsilon}_{UR} = X\left(\hat{\beta}_{UR} - \hat{\beta}_R\right)$. Since we know from class that $F_0 = \frac{\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}}{\hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}}\frac{n-k}{p}$, one possible place to start would be to solve for the expression $\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}$:

$$\begin{aligned}(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}) &= \hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_R - \hat{\varepsilon}'_R\hat{\varepsilon}_{UR} + \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}\\ &= \hat{\varepsilon}'_R\hat{\varepsilon}_R + \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR} - 2\hat{\varepsilon}'_{UR}\hat{\varepsilon}_R\end{aligned}$$

Where I used the fact that since $\hat{\varepsilon}'_{UR}\hat{\varepsilon}_R$ is a scalar, it is symmetric. Solving for $\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}$:

$$\begin{aligned}\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR} &= (\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}) + 2\hat{\varepsilon}'_{UR}\hat{\varepsilon}_R - 2\hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}\\ &= (\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}) + 2\hat{\varepsilon}'_{UR}(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})\\ &= (\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}) + 2\hat{\varepsilon}'_{UR}X\left(\hat{\beta}_{UR} - \hat{\beta}_R\right)\end{aligned}$$

But $\hat{\varepsilon}'_{UR}X = 0$ by construction. (Residuals are always orthogonal to regressors in OLS) Therefore,

$$\begin{aligned}\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR} &= (\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})\\ &= \left(\hat{\beta}_{UR} - \hat{\beta}_R\right)'X'X\left(\hat{\beta}_{UR} - \hat{\beta}_R\right)\end{aligned}$$

Substituting in the result from part (a) gives us:

$$\begin{aligned}\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR} &= \left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)'\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\Gamma(X'X)^{-1}(X'X)\cdot\\ &\quad (X'X)^{-1}\Gamma'\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)\\ &= \left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)'\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma(X'X)^{-1}\Gamma'\right)\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)\\ &= \left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)'\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)\end{aligned}$$

The right-hand side of this expression bears a striking resemblance to the F statistic:

$$\begin{aligned}F_0 &= \frac{1}{p}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)'\left(\hat{\sigma}^2\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)\\ &= \frac{1}{p}\frac{1}{\hat{\sigma}^2}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)'\left(\Gamma(X'X)^{-1}\Gamma'\right)^{-1}\left(\Gamma\hat{\beta}_{UR} - \gamma_0\right)\\ &= \frac{1}{p}\frac{1}{\hat{\sigma}^2}\left(\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}\right)\\ &= \frac{\left(\hat{\varepsilon}'_R\hat{\varepsilon}_R - \hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}\right)/p}{\hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}/(n-k)}\end{aligned}$$

Since $\hat{\sigma}^2 = \frac{\hat{\varepsilon}'_{UR}\hat{\varepsilon}_{UR}}{n-k}$.