W2003 Final

Q1 is from 1999 MT

Q2 is related to HW questions

Q4 is asymptotic normality / consistency of OLS

Q3 - Binary choice model

- migrate or not

$$u_{i,p} = x_{i,p}\beta_p + \varepsilon_{i,p} \quad \text{if stay}$$
$$u_{i,m} = x_{i,m}\beta_m + \varepsilon_{i,m} \quad \text{if migrate}$$
$$c_i = z_i\gamma + u_i \quad \text{(cost)}$$

} Want to estimate β_p, β_m, γ

Let $y_i = \begin{cases} 1 & \text{if stay} \\ 0 & \text{if migrate} \end{cases}$

Will stay iff $u_{i,p} \geq u_{i,m} - c_i$

iff $x_{i,p}\beta_p + \varepsilon_{i,p} \geq x_{i,m}\beta_m + \varepsilon_{i,m} - z_i\gamma - u_i$

iff $x_{i,p}\beta_p - x_{i,m}\beta_m + z_i\gamma \geq -\varepsilon_{i,p} + \varepsilon_{i,m} - u_i$

By assumption, $\begin{bmatrix} \varepsilon_{i,p} \\ \varepsilon_{i,m} \\ u_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \Sigma\right)$ where $\Sigma = \begin{bmatrix} \sigma_{pp} & \sigma_{pm} & \sigma_{pu} \\ & \sigma_{mm} & \sigma_{mu} \\ & & \sigma_{uu} \end{bmatrix}$

$$\Rightarrow [-1 \;\; 1 \;\; -1]\begin{bmatrix} \varepsilon_{i,p} \\ \varepsilon_{i,m} \\ u_i \end{bmatrix} = \underbrace{-\varepsilon_{i,p} + \varepsilon_{i,m} - u_i}_{\equiv \eta_i} \sim N\Big(0, \underbrace{[-1 \; 1 \; -1]\,\Sigma \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}}_{\Sigma^*}\Big)$$

where
$$\Sigma^* = [-1 \;\; 1 \;\; -1]\begin{bmatrix} \sigma_{pp} & \sigma_{pm} & \sigma_{pu} \\ \sigma_{pm} & \sigma_{mm} & \sigma_{mu} \\ \sigma_{pu} & \sigma_{mu} & \sigma_{uu} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$
$$= [-\sigma_{pp} + \sigma_{pm} - \sigma_{pu} \;/\; -\sigma_{pm} + \sigma_{mm} - \sigma_{mu} \;/\; -\sigma_{pu} + \sigma_{mu} - \sigma_{uu}]\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$
$$= \sigma_{pp} - \sigma_{pm} + \sigma_{pu} - \sigma_{pm} + \sigma_{mm} - \sigma_{mu} - \sigma_{pu} - \sigma_{mu} + \sigma_{uu}$$
$$= \sigma_{pp} - 2\sigma_{pm} - 2\sigma_{mu} + \sigma_{mm} + \sigma_{uu} \equiv \sigma^2$$

$$E[Y_i \mid X_{i,p}, X_{i,m}, Z_i] = Pr[Y_i = 1 \mid X_{i,p}, X_{i,m}, Z_i]$$
$$= Pr[\eta_i < X_{ip}\beta_p - X_{im}\beta_m + Z_i\gamma]$$
$$= Pr\left[\frac{\eta_i}{\sigma} < \frac{X_{ip}\beta_p - X_{im}\beta_m + Z_i\gamma}{\sigma}\right]$$
$$= \Phi\left(\frac{X_{ip}\beta_p - X_{im}\beta_m + Z_i\gamma}{\sigma}\right)$$

a) Probit model is

$$Y_i = \Phi\left(\frac{X_{ip}\beta_p - X_{im}\beta_m + Z_i\gamma}{\sigma}\right) + v_i = \Phi\left(\frac{W_i\delta}{\sigma}\right) + v_i$$

where $W_i = [\, X_{ip} \;\; -X_{im} \;\; Z_i \,]$, $\delta = \begin{bmatrix} \beta_p \\ \beta_m \\ \gamma \end{bmatrix}$

and $v_i = Y_i - E[Y_i \mid W_i]$

b) Construct the log-likelihood:

$$f(y_i \mid W_i) = \left[\Phi\left(\frac{W_i\delta}{\sigma}\right)\right]^{y_i}\left[1 - \Phi\left(\frac{W_i\delta}{\sigma}\right)\right]^{1-y_i}$$

$$\log L = \sum_{i=1}^{n} y_i \log \Phi\left(\frac{W_i\delta}{\sigma}\right) + \sum_{i=1}^{n}(1-y_i)\log\left[1 - \Phi\left(\frac{W_i\delta}{\sigma}\right)\right]$$

c) What is $\hat{\delta}^{MLE}$?

FOC:

$$(\delta): 0 = \sum_{i=1}^{n} y_i \frac{1}{\Phi\left(\frac{W_i\hat{\delta}^{MLE}}{\sigma}\right)}\varphi\left(\frac{W_i\hat{\delta}^{MLE}}{\sigma}\right)\cdot\frac{W_i'}{\sigma} + \sum_{i=1}^{n}(1-y_i)\frac{1}{1-\Phi\left(\frac{W_i\hat{\delta}^{MLE}}{\sigma}\right)}\left(-\varphi\left(\frac{W_i\hat{\delta}^{MLE}}{\sigma}\right)\right)\frac{W_i'}{\sigma}$$

$\hat{\delta}^{MLE}$ solves (δ).

But we might have an identification problem (if we don't know σ): we are estimating $\frac{\delta}{\sigma}$.

Weighted nonlinear least squares

Recall: $v_i = Y_i - E[Y_i | W_i]$

$$= \begin{cases} 1 - E[Y_i | W_i] & \text{if } Y_i = 1 \text{ w/prob } \Phi\left(\frac{w_i\delta}{\sigma}\right) \\ -E[Y_i | W_i] & \text{if } Y_i = 0 \text{ w/prob } 1 - \Phi\left(\frac{w_i\delta}{\sigma}\right) \end{cases}$$

$$\Rightarrow V(v_i | W_i) = \Phi\left(\frac{w_i\delta}{\sigma}\right)\left[1 - \Phi\left(\frac{w_i\delta}{\sigma}\right)\right]$$

Thus, we have

$$\hat{\delta}^{WNLS} = \arg\min_{\delta} \sum_{i=1}^{n} \frac{\{Y_i - \Phi\left(\frac{w_i\delta}{\sigma}\right)\}^2}{\Phi\left(\frac{w_i\delta_0}{\sigma}\right)\left[1 - \Phi\left(\frac{w_i\delta_0}{\sigma}\right)\right]}$$

FOC:

$$(\delta): 0 = 2\sum_{i=1}^{n} \frac{\{Y_i - \Phi\left(\frac{w_i\hat{\delta}^{WNLS}}{\sigma}\right)\}}{\Phi\left(\frac{w_i\delta_0}{\sigma}\right)\left[1 - \Phi\left(\frac{w_i\delta_0}{\sigma}\right)\right]}\left(-\varphi\left(\frac{w_i\hat{\delta}^{WNLS}}{\sigma}\right)\right)\frac{w_i'}{\sigma}$$

$\hat{\delta}^{WNLS}$ solves (δ)

d) Discuss the asymptotic properties of these estimators

- Under regularity conditions, ML and WNLS are asymptotically normal and consistent.

$$\sqrt{n}\left(\hat{\delta}_{ML} - \delta_0\right) \xrightarrow{d} N\left(0, [I_1(\delta_0)]^{-1}\right)$$

where $I_1(\delta_0) = \frac{1}{n}\underbrace{E\left[-\frac{\partial^2 \log L}{\partial\delta\,\partial\delta'}\Big|_{\delta_0}\right]}_{= I_n(\delta_0)} = -E\left[\frac{\partial}{\partial\delta'}\underbrace{\left[\frac{-(Y_i - \Phi\left(\frac{w_i\delta}{\sigma}\right))}{\Phi\left(\frac{w_i\delta}{\sigma}\right)\left[1 - \Phi\left(\frac{w_i\delta}{\sigma}\right)\right]}\varphi\left(\frac{w_i\delta}{\sigma}\right)\frac{w_i'}{\sigma}\right]}_{s(w_i,\delta)}\Bigg|_{\delta_0}\right]$

$$= E\left[s(w_i,\delta_0)\, s(w_i,\delta_0)'\right]$$

$$= E\left[\left\{\frac{Y_i - \Phi\left(\frac{w_i\delta_0}{\sigma}\right)}{\Phi\left(\frac{w_i\delta_0}{\sigma}\right)\left[1 - \Phi\left(\frac{w_i\delta_0}{\sigma}\right)\right]}\right\}^2 w_i' w_i\right]$$