Review session on Tuesday; March 21st at 10 am.

Final 1999, Q2:

Omitted variable bias

True model: $Y^L = X\beta + Z\gamma + \varepsilon = [X \;\; Z]\begin{bmatrix}\beta \\ \gamma\end{bmatrix} + \varepsilon$

Estimated model: $Y^S = X\beta + u$ where $u = (Z\gamma + \varepsilon)$

Estimate: $\hat{\beta} = (X'X)^{-1}X'Y$

Is $\hat{\beta}$ biased? consistent? What is the variance?

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + Z\gamma + \varepsilon)$$

$$= \beta + (X'X)^{-1}X'Z\gamma + (X'X)^{-1}X'\varepsilon$$

$$E[\hat{\beta} \mid X, Z] = \beta + \underbrace{(X'X)^{-1}X'E[\varepsilon \mid X, Z]}_{=0} + (X'X)^{-1}X'Z\gamma$$

$$= \beta + (X'X)^{-1}X'Z\gamma = \beta$$

iff $\gamma = 0$ or $X'Z = 0$

In general, though, there is bias

Consistency: $\hat{\beta} = \beta + \underbrace{\left(\frac{1}{n}\sum X_i'X_i\right)^{-1}}_{\xrightarrow{p} E[X_i'X_i]^{-1}}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_i'z_i\right)\gamma}_{\xrightarrow{p} E[X_i'z_i]\gamma} + \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_i'X_i\right)}_{\xrightarrow{p} E[X_i'X_i]}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_i'\varepsilon_i\right)}_{\xrightarrow{p} E[X_i'\varepsilon_i]=0}$

$$\xrightarrow{p} \beta + E[X_i'X_i]^{-1}E[X_i'z_i]\gamma = \beta$$

iff $\gamma = 0$ or $E[X_i'z_i] = 0$

In general, though, the estimator is not consistent.

$$V(\hat{\beta}^s \mid X, Z) = V((X'X)^{-1}X'(X\beta + u) \mid X, Z)$$

$$= V((X'X)^{-1}X'u \mid X, Z)$$

$$= (X'X)^{-1}X'V(u)X(X'X)^{-1}$$

$$= (X'X)^{-1}X'V(Z\gamma + \varepsilon \mid X, Z)X(X'X)^{-1}$$

$$= (X'X)^{-1}X'V(\varepsilon \mid X, Z)X(X'X)^{-1}$$

$$= \sigma^2 (X'X)^{-1}$$

$$\hat{\beta}^L = (X' M_Z X)^{-1} X' M_Z Y = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'(X\beta + Z\gamma + \varepsilon)$$

$$\Rightarrow V(\hat{\beta}^L | X, Z) = V((\tilde{X}'\tilde{X})^{-1}\tilde{X}'\varepsilon | X, Z)$$

$$= (\tilde{X}'\tilde{X})^{-1}\tilde{X}' V(\varepsilon | X, Z)\tilde{X}(\tilde{X}'\tilde{X})^{-1}$$

$$= \sigma^2(\tilde{X}'\tilde{X})^{-1}$$

$$= \sigma^2(X' M_Z X)^{-1}$$

$$V(\hat{\beta}^L | X, Z) - V(\hat{\beta}^S | X, Z) = \sigma^2 (X'X)^{-1} - \sigma^2 (X' M_Z X)^{-1} \doteq 0$$

$$\text{iff} \quad V(\hat{\beta}^L | X, Z)^{-1} - V(\hat{\beta}^S | X, Z)^{-1} = \frac{1}{\sigma^2}(X' M_Z X - X'X)$$

$$= -\underbrace{\frac{1}{\sigma^2}}_{\geq 0}\underbrace{(X' P_Z X)}_{psd} \leq 0, \text{ which is always true.}$$

Thus, in matrix sense, $V(\hat{\beta}^L | X, Z) \geq V(\hat{\beta}^S | X, Z)$

Final 2000, Question 1B

Measurement error:

$Y_i^* = \alpha + \beta X_i^* + \varepsilon_i$ but we only observe $Y_i = Y_i^* + v_i$.
$X_i = X_i^*$

- Is the estimator consistent?
  - assume that $E[\varepsilon_i] = E[v_i] = E[v_i X_i^*] = E[v_i Y_i^*] = E[v_i \varepsilon_i] = 0$

$$Y_i + v_i = \alpha + \beta X_i + \varepsilon_i$$

$$\Rightarrow Y_i = \alpha + \beta X_i + \underbrace{(\varepsilon_i - v_i)}_{\equiv u_i}$$

$$\Rightarrow Y_i = Z_i \theta + u_i \qquad Z_i = [1 \ X_i], \ \theta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\hat{\theta} = (Z'Z)^{-1}Z'Y = \left(\frac{1}{n}\sum_{i=1}^{n} z_i' z_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' Y_i\right)$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} z_i' z_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i'(z_i\theta + u_i)\right)$$

$$= \theta + \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} z_i' z_i\right)^{-1}}_{\xrightarrow{P} (E[z_i' z_i])^{-1}}\left(\frac{1}{n}\sum_{i=1}^{n} z_i' u_i\right)$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} z_i' u_i\right) = \left(\frac{1}{n}\sum_{i=1}^{n} z_i' \varepsilon_i\right) + \left(\frac{1}{n}\sum_{i=1}^{n} z_i' v_i\right)$$

$$\xrightarrow{P} E[z_i'\varepsilon_i] = E\begin{bmatrix}\varepsilon_i \\ X_i'\varepsilon_i\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad \xrightarrow{P} E[z_i' v_i] = E\begin{bmatrix} v_i \\ X_i'\varepsilon_i\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Need to argue this rigorously

$\hat{\theta} \xrightarrow{P} \theta.$

Final 2004, Q1

1] $Y_i = X_i\beta + \varepsilon_i \Rightarrow \hat{\beta} = \bar{Y}$? T/F.

$$\hat{\beta} = (X'X)^{-1}X'Y = \frac{\sum X_i Y_i}{\sum X_i^2} \neq \frac{1}{n}\sum_{i=1}^{n} Y_i \text{ unless } X_i = 1\ \forall i.$$

False!

2] $Y_i = X_i\beta + Z_i\gamma$