No asymptotics on the midterm (next Thursday)

asymptotics: What happens as sample size goes to infinity?

$$\hat{\beta}_{OLS} = \underbrace{(X'X)^{-1}X'Y}_{1\text{st } 5 \text{ weeks}} = \underbrace{\left(\sum_{i=1}^{n} X_i'X_i\right)^{-1} \sum_{i=1}^{n} X_i'Y_i}_{\text{final } 5 \text{ weeks}} \leftarrow \text{This is a fcn of } n$$

$\Rightarrow$ use notation $\hat{\beta}_n$. What happens to $\hat{\beta}_n$ as $n \to \infty$?

Defn: Let $\{a_n\}_{n=1}^{\infty} \in \ell^{\infty}$. We say that $a_n \to a$ if $\forall \varepsilon > 0$ $\exists N(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.

Defn: Let $\{f_n\}_{n=1}^{\infty}$, $f_n : \mathbb{R} \to \mathbb{R}$ be s.t. $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in \mathbb{R}$. Then we say that $f_n \to f$ pointwise.

In probability, we have 4 different topologies:

Almost sure convergence

$$P\left(\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1 \iff X_n \xrightarrow{a.s.} X$$
$$\iff X_n - X \xrightarrow{a.s.} 0$$

Convergence in probability

$$\lim_{n\to\infty} P(\omega : \|X_n(\omega) - X(\omega)\| \leq \varepsilon) = 1 \iff X_n \xrightarrow{P} X$$
$$= \lim_{n\to\infty} Pr(|X_n(\omega) - X(\omega)| \leq \varepsilon) = 1 \quad \text{(univariate case)}$$

a.s. $\Rightarrow$ p but p $\not\Rightarrow$ a.s.

Defn: If $\hat{\theta}_n \xrightarrow{a.s.} \theta$, then $\hat{\theta}_n$ is strongly consistent.

Defn: If $\hat{\theta}_n \xrightarrow{P} \theta$, then $\hat{\theta}_n$ is (weakly) consistent.

Thm 6.1.ii: If $X_n \xrightarrow{P} X$ and g is continuous a.e., then $g(X_n) \xrightarrow{P} g(X)$. If $X_n \xrightarrow{a.s.} X$, then $g(X_n) \xrightarrow{a.s.} g(X)$.

Schemes for data:

iid	indep.	ident. distr.
inid	indep.	not indent. distr.
niid	not indep.	ident. distr.
ninid	not indep	not indent. distr.

Here, we are only interested in LLN's for iid and inid schemes.

Laws of Large Numbers: (Strong, Weak)

Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \mu_i$ where $\mu_i \equiv E[X_i]$ is true mean.

(Kolmogorov)
iid LLN: Let $\{X_i\}$ be iid. Then $\bar{X}_n \xrightarrow{a.s.} E[X_i] \equiv \mu$

$\Leftrightarrow \bar{X}_n - \mu \xrightarrow{a.s.} 0 \Leftrightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow{a.s.} 0$

Proposition: Let g be a measurable function. If X_i and X_j are identically distributed, then $g(X_i)$ and $g(X_j)$ are identically distributed. If X_i and X_j are independent, then $g(X_i)$ and $g(X_j)$ are independent.

(Markov)
inid LLN: Let $\{X_i\}$ be independent with $\mu_i \equiv E[X_i] < +\infty \ \forall i$.

If for $\delta > 0$, $\sum_{i=1}^{\infty} \frac{E[|X_i - \mu_i|^{1+\delta}]}{i^{1+\delta}} < +\infty$, then $\bar{X}_n - \bar{\mu}_n \xrightarrow{a.s.} 0$

Consistency of OLS

We are no longer in the classical model. New assumptions:

2.1. $Y_i = X_i\beta + \varepsilon_i$

2.2 $\{(Y_i, X_i)\}_{i=1}^{n}$ iid $\Leftrightarrow \{(X_i, \varepsilon_i)\}_{i=1}^{n}$ iid

2.3 (a) $E[X_i' \varepsilon_i] = 0$ X_i is uncorrelated with ε_i
(*) contemporaneous exogeneity

(b) $E[|X_{ik}\varepsilon_i|] < +\infty$ $\forall i$ $\forall k$

2.4 (a) $E[|X_{ik}|^2] < +\infty \quad \forall i=1,\dots,n \quad \forall k$

(b) $E[\underset{k\times n}{X_i'}\underset{n\times k}{X_i}]$ is positive definite

Here, we are assuming random sampling.

$$\hat{\beta}_n = \left(\frac{1}{n}\sum_{i=1}^{n} X_i' X_i\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} X_i' X_i\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i'(X_i\beta + \varepsilon_i)$$

$$= \beta + \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_i' X_i\right)^{-1}}_{\xrightarrow{as.} (E[X_i'X_i])^{-1}} \underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i' \varepsilon_i}_{\xrightarrow{a.s.} E[X_i'\varepsilon_i] = 0}$$

by Kolmogorov by continuity thm by assumption

$$\Rightarrow \hat{\beta}_n \xrightarrow{as} \beta + 0 = \beta$$

Therefore $\hat{\beta}_{OLS}$ is a consistent estimator