$$t_0 \equiv \frac{\Gamma\hat{\beta} - \Gamma\beta_0}{\sqrt{\hat{\sigma}^2 \Gamma (X'X)^{-1}\Gamma'}} = \frac{\Gamma\hat{\beta} - \Gamma\beta_0}{se} \sim t(n-k) \quad \text{under } H_0 : \Gamma\beta = \Gamma\beta_0$$

vs $H_1 : \Gamma\beta \neq \Gamma\beta_0$

Let $\Gamma\beta_0 = 0$, $\Gamma = [0 \cdots 010 \cdots 0]$

$$t_0 = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)} \sim t(n-k) \quad \text{under } H_0$$

Under H_1, however,

$$t_0 = \underbrace{\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)}}_{\sim t(n-k)} + \underbrace{\frac{\beta_j}{se(\hat{\beta}_j)}}_{?} = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}$$

How does t_0 behave?

$$se(\hat{\beta}_j) = \sqrt{\hat{\sigma}^2 \Gamma (X'X)^{-1}\Gamma'} = \sqrt{\hat{\sigma}^2 \Gamma \left(\tfrac{1}{n}X'X\right)^{-1} \cdot \tfrac{1}{n}}$$

$$\frac{1}{n} X'X = \frac{1}{n}\sum_{i=1}^{n} x_i' x_i \xrightarrow[n \to \infty]{p} E[x_i' x_i] \quad \text{by WLLN}$$

$$\Rightarrow \frac{\beta_j}{se(\hat{\beta}_j)} = \frac{\sqrt{n}\,\beta_j}{\underbrace{\sqrt{\hat{\sigma}^2 \Gamma \left(\tfrac{1}{n}X'X\right)^{-1}\Gamma'}}_{\to \text{constant}}} \longrightarrow \pm\infty$$

Thus, under H_1, $t_0 \to \pm\infty$. This is why the t-test is consistent. (Explains why we can reject the null when the null is not true).

Restricted Least Squares

MT 2000, Q4)

unrestricted model. $Y = X\beta + \varepsilon \Rightarrow \hat{\beta}_{OLS} = (X'X)^{-1}X'Y$

restricted model: $Y = X\beta + \varepsilon$, $\underset{p\times k}{\Gamma}\beta = \underset{p\times 1}{\gamma_0}$

e.g. $\beta_1 = 0 \iff \Gamma = [1\ 0 \cdots 0]_{1\times k}$, $\gamma_0 = 0$

Constrained optimization:

$$\tilde{\beta} = \hat{\beta}_R \equiv \underset{b:\Gamma b=\gamma_0}{\operatorname{argmin}} \sum_{i=1}^{n} (Y_i - X_i b)^2 \overset{\text{monotonic transf. of obj. fcn.}}{=} \underset{b:\Gamma b=\gamma_0}{\operatorname{argmin}} \frac{1}{2} \sum_{i=1}^{n} (Y_i - X_i b)^2$$

$$\Rightarrow \min \frac{1}{2} (Y - Xb)'(Y - Xb) + \underset{1\times p}{\lambda} (\underset{p\times k}{\Gamma} \underset{k\times 1}{b} - \gamma_0)$$

$$\Leftrightarrow \min \frac{1}{2} (Y'Y - Y'Xb - b'X'Y + b'X'Xb) + \lambda(\Gamma\beta - \gamma_0)$$

FOCs:

$$(b): \frac{1}{2}(-2X'Y + 2X'X\hat{\beta}_R) + \Gamma'\lambda' = 0$$

$$\Leftrightarrow \frac{1}{2}(-2X'Y + 2X'X\hat{\beta}_R) = -\Gamma'\lambda'$$

$$\Leftrightarrow X'Y - X'X\hat{\beta}_R = \underset{k\times p}{\Gamma'} \underset{p\times 1}{\lambda'}$$

$$\Leftrightarrow X'X\hat{\beta}_R = X'Y - \Gamma'\lambda'$$

$$\Leftrightarrow \hat{\beta}_R = (X'X)^{-1}X'Y - (X'X)^{-1}\Gamma'\lambda'$$

$$= \hat{\beta}_{UR} - (X'X)^{-1}\Gamma'\lambda' \qquad (1)$$

also, $\Gamma\hat{\beta}_R - \gamma_0 = 0$

$$\Rightarrow \Gamma\hat{\beta}_{UR} - \Gamma(X'X)^{-1}\Gamma'\lambda' - \gamma_0 = 0$$

$$\Leftrightarrow \lambda' = (\Gamma(X'X)^{-1}\Gamma')^{-1}(\Gamma\hat{\beta}_{UR} - \gamma_0) \qquad (2)$$

Substitute (2) into (1):

$$\hat{\beta}_R = \hat{\beta}_{UR} - (X'X)^{-1}\Gamma'(\Gamma(X'X)^{-1}\Gamma')^{-1}(\Gamma\hat{\beta}_{UR} - \gamma_0)$$

Examples:

1] Suppose $\Gamma = I_k$, $\gamma_0 = 0$

$$\Rightarrow \hat{\beta}_R = \hat{\beta}_{UR} - (X'X)^{-1}X'X(\hat{\beta}_{UR}) = 0$$

2] Suppose $\Gamma = \gamma_0 = 0$

$$\Rightarrow \hat{\beta}_R = \hat{\beta}_{UR} - (X'X)^{-1}0\,(0(X'X)^{-1}0)^{-1}(0\hat{\beta}_{UR} - 0) = \hat{\beta}_{UR}$$

b) $\hat{\varepsilon}_R \equiv \underline{Y} - \underline{X}\hat{\beta}_R$

$\hat{\varepsilon}_{UR} \equiv \underline{Y} - \underline{X}\hat{\beta}_{UR}$

$\Rightarrow \quad \hat{\varepsilon}_R - \hat{\varepsilon}_{UR} = \underline{X}(\hat{\beta}_{UR} - \hat{\beta}_R) \qquad (1)$

also, $\hat{\varepsilon}_R'\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR} = (\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}) + 2\hat{\varepsilon}_{UR}'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})$

From (1): $= (\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}) + 2\underbrace{\hat{\varepsilon}_{UR}'\underline{X}}_{=0 \text{ by orthogonality}}(\hat{\beta}_{UR} - \hat{\beta}_R)$

$= (\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})'(\hat{\varepsilon}_R - \hat{\varepsilon}_{UR})$

$= (\hat{\beta}_{UR} - \hat{\beta}_R)'\underline{X}'\underline{X}(\hat{\beta}_{UR} - \hat{\beta}_R)$

$\Rightarrow$ From part (a):

$$\hat{\varepsilon}_R'\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR} = (\Gamma\hat{\beta}_{UR} - \gamma_0)'(\Gamma(\underline{X}'\underline{X})^{-1}\Gamma')^{-1}\Gamma(\underline{X}'\underline{X})^{-1}(\underline{X}'\underline{X}) \cdot (\underline{X}'\underline{X})^{-1}\Gamma'(\Gamma(\underline{X}'\underline{X})^{-1}\Gamma')^{-1}(\Gamma\hat{\beta}_{UR} - \gamma_0)$$

$$= (\Gamma\hat{\beta}_{UR} - \gamma_0)'(\Gamma(\underline{X}'\underline{X})^{-1}\Gamma')^{-1}(\Gamma\hat{\beta}_{UR} - \gamma_0)$$

$$= \underbrace{\frac{\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}}{n-k}}_{\hat{\sigma}^2} \cdot pF_0$$

$$\Rightarrow \quad F_0 = \frac{(\hat{\varepsilon}_R'\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR})/p}{\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}/(n-k)}$$

Remark: $F_0 = \dfrac{(R^2_{UR} - R^2_R)/p}{(1 - R^2_{UR})/(n-k)}$ since $R^2_{UR} = 1 - \dfrac{\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}}{\underline{Y}'M^0\underline{Y}}$

$\Leftrightarrow \quad \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR} = (1 - R^2_{UR})(\underline{Y}'M_0\underline{Y})$

STATA

$$se = \sqrt{\hat{\sigma}^2 \Gamma (X'X)^{-1} \Gamma'}$$

$$educ \Rightarrow \Gamma = [1 \quad 0 \quad 0 \quad 0]$$

$$t_0 \equiv \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}$$

For a hypothesis test of $\Gamma \beta = 0$, compare $|t_0|$ and $c^*_{0.05,\, t(n-k)}$

What is the $P>|t|$ term?

Let $T \sim t(n-k)$. Then $(P>|t|) \equiv 2\Pr[T>|t_0|]$ where t_0 is the realization. (i.e. is a constant in this calculation)

If $(P>|t|) < 0.05$, then we reject the null and conclude statistical significance.