Today
- Idempotent matrices
- Partitioned regressions
- CNLR
- R2

Idempotent Matrices

1] Projection matrix

$$\underset{n\times k}{P_X} \equiv X\,(\underset{k\times n}{X'}\,\underset{n\times k}{X})^{-1}\underset{k\times n}{X'}$$

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y = P_X Y$$ - motivation for calling P_X the projection matrix.

Rewriting the model $Y = X\beta + \varepsilon$ as

$$Y = X_1\beta_1 + X_2\beta_2 + \varepsilon$$

Here, $P_{X_1}X_2 = X_1(X_1'X_1)^{-1}X_1 X_2 = \hat{X}_2$

The projection matrix P_X is idempotent:

$$P_X P_X = X(X'X)^{-1}\underbrace{X'X(X'X)}_{I_k}{}^{-1}X' = X(X'X)^{-1}X' = P_X$$

Also, P_X is symmetric:

$$P_X' = [X(X'X)^{-1}X']' = [(X'X)^{-1}X']'X' = (X')'[(X'X)^{-1}]'X'$$
$$= X([X'X]')^{-1}X' = X(X'(X')')^{-1}X' = X(X'X)^{-1}X' = P_X$$

2] Annihilator matrix

$M_X \equiv I_n - P_X$

$M_X X = (I_n - P_X)X = X - P_X X = X - X = 0$ annihilates X

$M_X Y = (I_n - P_X)Y = Y - P_X Y = Y - \hat{Y} = \hat{\varepsilon}$ Gives us the residual

Wrt the partitioned regression:

$$M_{X_1}X_2 = \underset{\text{residual}}{\hat{\eta}} \text{ if } X_2 = \gamma X_1 + \eta$$

It is easy to create the M matrix in Matlab $\Rightarrow$ very useful

$$M_X M_X = (I_n - P_X)(I_n - P_X) = I_n - P_X - P_X + \underbrace{P_X P_X}_{P_X} = I - P_X - P_X + P_X = I - P_X$$
$$= M_X$$

Partitioned regression

$$Y = \underset{n\times k}{X}\underset{k\times 1}{\beta} + \varepsilon$$

$$= \underset{n\times k_1}{X_1}\underset{k_1\times 1}{\beta_1} + \underset{n\times k_2}{X_2}\underset{k_2\times 1}{\beta_2} + \varepsilon \qquad k_1+k_2=k$$

$$\underset{n\times k}{X} = [\underset{n\times k_1}{X_1} \quad \underset{n\times k_2}{X_2}] \qquad \underset{k\times n}{X'} = \begin{bmatrix} \underset{k_1\times n}{X_1'} \\ \underset{k_2\times n}{X_2'} \end{bmatrix}$$

$$\Rightarrow \hat\beta = (X'X)^{-1}X'Y$$

$$= \left(\begin{bmatrix} X_1' \\ X_2' \end{bmatrix}[X_1 \quad X_2]\right)^{-1}\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} Y$$

$$\underset{k=k_1+k_2}{=} \Bigg\{ \underbrace{\begin{bmatrix} \underset{k_1\times n}{X_1'}\underset{n\times k_1}{X_1} & \underset{k_1\times n}{X_1'}\underset{n\times k_2}{X_2} \\ \underset{k_2\times n}{X_2'}\underset{n\times k_1}{X_1} & \underset{k_2\times n}{X_2'}\underset{n\times k_2}{X_2} \end{bmatrix}^{-1}}_{k=k_1+k_2} \underbrace{\begin{bmatrix} \underset{k_1\times n}{X_1'}\underset{n\times 1}{Y} \\ \underset{k_2\times n}{X_2'}\underset{n\times 1}{Y} \end{bmatrix}}_{k=k_1+k_2}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$$

Lemma: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & F^{-1} \end{bmatrix}$ where $E = A - BD^{-1}C$, $F = D - CA^{-1}B$

This gives us: $\hat\beta = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$ where $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}$

$$= \begin{bmatrix} A_1X_1'Y + B_1X_2'Y \\ C_1X_1'Y + D_1X_2'Y \end{bmatrix} \equiv \begin{bmatrix} \hat\beta_1 \\ \hat\beta_2 \end{bmatrix}$$

Substituting, $\hat\beta_1 = (A - BD^{-1}C)X_1'Y - (A-BD^{-1}C)^{-1}BD^{-1}X_2'Y$

$$= (X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1)^{-1}X_1'Y$$
$$- (X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1)^{-1}X_1'X_2(X_2'X_2)^{-1}X_2'Y$$

$$= (X_1'X_1 - X_1'P_{X_2}X_1)^{-1}(X_1' - X_1'P_{X_2})Y$$

$$= [X_1'(X_1 - P_{X_2}X_1)]^{-1}(X_1' - X_1'P_{X_2})Y$$

$$= [X_1'(I_n - P_{X_2})X_1]^{-1}X_1'(I_n - P_{X_2})Y$$

$$= [X_1'M_{X_2}X_1]^{-1}X_1'M_{X_2}Y$$

$$= [X_1'M_{X_2}'M_{X_2}X_1]^{-1}X_1'M_{X_2}'M_{X_2}Y$$

$$\equiv (\tilde X_1'\tilde X_1)^{-1}\tilde X_1'\tilde Y \qquad \text{where } \tilde X_1 = M_{X_2}X_1$$

Similarly, $\hat{\beta}_2 = (\tilde{X}_2' \tilde{X}_2)^{-1} \tilde{X}_2' Y$ where $\tilde{X}_2 = M_{X_1} X_2$

This is the estimator of $Y = \underbrace{\tilde{X}_2'}_{\text{The part of } X_2 \text{ that cannot be explained by } X_1} \beta_2 + \varepsilon$

$$\hat{\beta}_1 = (X_1' M_{X_2} X_1)^{-1} X_1' M_{X_2} Y = (\tilde{X}_1' \tilde{X}_1)^{-1} \tilde{X}_1' Y$$

What is $V(\hat{\beta}_1 | X)$?

$$V(\hat{\beta}_1 | X) = V((\tilde{X}_1' \tilde{X}_1)^{-1} \tilde{X}_1' Y | X)$$

$$= (\tilde{X}_1' \tilde{X}_1)^{-1} \tilde{X}_1' \underbrace{V(Y|X)}_{\sigma^2 I_n} \tilde{X}_1 (\tilde{X}_1' \tilde{X}_1)^{-1}$$

$$= \sigma^2 (\tilde{X}_1' \tilde{X}_1)^{-1} \tilde{X}_1' \tilde{X}_1 (\tilde{X}_1' \tilde{X}_1)^{-1}$$

$$= \sigma^2 (\tilde{X}_1' \tilde{X}_1)^{-1} = \sigma^2 (X_1' M_{X_2} X_1)^{-1}$$

$V(\hat{\beta}_1 | X) \uparrow$ as $\tilde{X}_1' \tilde{X}_1 \downarrow$