wts: $\frac{1}{n}\sum \hat{\varepsilon}_i^4 \xrightarrow{p} E[\varepsilon_i^4]$

note: $\frac{1}{n}\sum \hat{\varepsilon}_i^4 - E[\varepsilon_i^4] = \frac{1}{n}\sum \hat{\varepsilon}_i^4 - \varepsilon_i^4 + \left(\frac{1}{n}\sum \varepsilon_i^4 - E[\varepsilon_i^4]\right)$

0 by WLLN under some conditions

next:

$$
\begin{aligned}
\hat{\varepsilon}^4 - \varepsilon^4 &= (\hat{\varepsilon}^2)^2 - (\varepsilon^2)^2 \\
&= (\hat{\varepsilon}^2 + \varepsilon^2)(\hat{\varepsilon}^2 - \varepsilon^2) \\
&= \hat{\varepsilon}^2(\hat{\varepsilon}^2 - \varepsilon^2) + \varepsilon^2(\hat{\varepsilon}^2 - \varepsilon^2) \\
&= \hat{\varepsilon}^2(\hat{\varepsilon}^2 - \varepsilon^2) + 2\varepsilon^2(\hat{\varepsilon}^2 - \varepsilon^2) - \varepsilon^2(\hat{\varepsilon}^2 - \varepsilon^2) \\
&= \underbrace{(\hat{\varepsilon}^2 - \varepsilon^2)^2}_{A} + \underbrace{2\varepsilon^2(\hat{\varepsilon}^2 - \varepsilon^2)}_{B}
\end{aligned}
$$

B:

$$
\begin{aligned}
\frac{1}{n}\sum \varepsilon_i^2(\hat{\varepsilon}_i^2 - \varepsilon_i^2) &= \frac{1}{n}\sum \varepsilon_i^2\left[(\hat{\varepsilon}_i - \varepsilon_i)^2 + 2\varepsilon_i(\hat{\varepsilon}_i - \varepsilon_i)\right] \\
&= \frac{1}{n}\sum \varepsilon_i^2[x_i'(\beta-\hat{\beta})]^2 + 2\varepsilon_i[x_i(\beta-\hat{\beta})] \\
&= (\beta-\hat{\beta})'\left(\frac{1}{n}\sum \varepsilon_i^2 x_i' x_i\right)(\hat{\beta}-\beta) + 2\left(\frac{1}{n}\sum \varepsilon_i x_i\right)(\beta-\hat{\beta}) \\
&\xrightarrow{p} 0 \times E[\varepsilon_i^2 x_i' x_i] \times 0 + 2 \times 0 \times 0
\end{aligned}
$$

if $E[\varepsilon_i^2 x_i' x_i] < \infty$

and b/c $E[\varepsilon_i x_i] = 0$

A:

$$
\begin{aligned}
0 \le \frac{1}{n}\sum(\hat{\varepsilon}_i^2 - \varepsilon_i^2)^2 &= \frac{1}{n}\sum\left((\hat{\varepsilon}_i - \varepsilon_i)^2 + 2\varepsilon_i(\hat{\varepsilon}_i - \varepsilon_i)\right)^2 \\
&= \frac{1}{n}\sum(\hat{\varepsilon}_i - \varepsilon_i)^4 + 4\varepsilon_i(\hat{\varepsilon}_i - \varepsilon_i)^3 + 4\varepsilon_i^2(\hat{\varepsilon}_i - \varepsilon_i)^2 \\
&= \frac{1}{n}\sum(x_i'(\beta-\hat{\beta}))^4 + 4\varepsilon_i(x_i'(\beta-\hat{\beta}))^3 + 4\varepsilon_i^2(x_i'(\beta-\hat{\beta}))^2
\end{aligned}
$$

Cauchy-Schwarz

$$
\begin{aligned}
&\le \frac{1}{n}\sum \|x_i\|^4\|\beta-\hat{\beta}\|^4 + 4\varepsilon_i\|x_i\|^3\|\beta-\hat{\beta}\|^3 + 4\varepsilon_i^2\|x_i\|^2\|\beta-\hat{\beta}\|^2 \\
&\xrightarrow{p} E\|x_i\|^4 \times 0 + 4E[\varepsilon_i\|x_i\|^3] \times 0 + 4E\|\varepsilon_i x_i\|^2 \times 0
\end{aligned}
$$

if $E\|x_i\|^4 < \infty$ and $E[\varepsilon_i\|x_i\|^3] < \infty$