Econ 203B: Single Equation Models

Matrix Definiteness

Michael Powell
Department of Economics, UCLA
February 10th, 2006

The definiteness of matrices comes up quite frequently in dealing with the finite sample properties of the OLS estimator. I have put together a short list of basic results for establishing that a matrix is positive semi-definite. I hope eventually to be able to provide a proof for proposition 6.

Definition 1 *An $n \times n$ matrix A is* ***positive semi-definite*** *if for any non-zero $n \times 1$ vector t, we have $t'At \geq 0$.*

Proposition 2 *For any $n \times k$ matrix X, $X'X$ is positive semi-definite.*

Proof. Let $t = \begin{bmatrix} t_1 \\ \vdots \\ t_k \end{bmatrix} \neq 0$. Then we have

$$\begin{aligned} t'X'Xt &= (Xt)'(Xt) \\ &= s's \\ &= \sum_{i=1}^{n} s_i^2 \geq 0 \end{aligned}$$

Where s is an $n \times 1$ vector defined by $Xt \equiv s$ ■

Proposition 3 *Let A be a positive semi-definite $k \times k$ matrix. Then A' is positive semi-definite.*

Proof. Let $t = \begin{bmatrix} t_1 \\ \vdots \\ t_k \end{bmatrix} \neq 0$. Then we have:

$$0 \leq t'At = (t'A't)'$$

But since $(t'A't)$ is a scalar, $(t'A't)' = t'A't$. Therefore, we have:

$$t'A't \geq 0$$

Giving us the result. ■

Proposition 4 *Let A be a positive semi-definite $k \times k$ non-singular matrix. Then A^{-1} is positive semi-definite.*

Proof. Let $t \neq 0$ be an arbitrary $k \times 1$ vector. Since A is nonsingular, for every $k \times 1$ vector t, there exists a $k \times 1$ vector s such that $As = t$. Also, since A is nonsingular and positive semi-definite, A' is also nonsingular and positive semi-definite. We have, then:

$$\begin{aligned} 0 &\leq s'A's = s'A'A^{-1}As \\ &= (As)'A^{-1}As = t'A^{-1}t \end{aligned}$$

Which establishes that A^{-1} is positive semi-definite. ■

Proposition 5 *Let X be an $n \times k$ matrix. The projection matrix $P_X = X \left(X'X\right)^{-1} X'$ is positive semi-definite.*

Proof. Let $t \neq 0$ be an arbitrary $n \times 1$ vector. Then, since $X'X$ is positive semi-definite by proposition 2 and thus $\left(X'X\right)^{-1}$ is positive semi-definite by proposition 3, we have:

$$\begin{aligned} t'P_X t &= t'X \left(X'X\right)^{-1} X't \\ &= \left(X't\right)' \left(X'X\right)^{-1} \left(X't\right) \\ &= s' \left(X'X\right)^{-1} s \geq 0 \end{aligned}$$

Where $s \equiv X't$ is a $k \times 1$ vector. ■

Proposition 6 *If A and B are non-singular, positive semi-definite, symmetric matrices, then $A - B$ is positive semi-definite if and only if $B^{-1} - A^{-1}$ is positive semi-definite. (Equivalently, $A - B$ is psd iff $A^{-1} - B^{-1}$ is nsd.)*