$$V(z,w,r) = \max_{\hat{a}_{t+1}} \left\{ u(z_t - \hat{a}_{t+1}) + \beta \int V(z_{t+1}, w, r)\, dG(l_{t+1}) \right\}$$

$\hat{a}_{t+1} \geq z_t$

$z_{t+1} = w l_{t+1} + (1+r)\hat{a}_{t+1} - r\varphi$

$z \in [w l_{min} - r\varphi, \infty]$

$w l_{min} - r\varphi = 0$ iff φ is natural borrowing limit

Suppose we solved and got the policy function $A(z,w,r)$

$A(z^*, w, r)(1+r) + w l_{max} = z^*$

$z \neq z^* \Rightarrow A(z,w,r)(1+r) + w l_{max} < z^*$

$[z_{min}, z^*]$ is the space in which peoples' asset holdings would lie

$u'(z - A(z)) = \beta E[V'(\cdots)]$

$$-u'(z - A(z))\frac{dA}{dz} + \beta\left[\int V'(wl' + (1+r)A(z) - r\varphi)\, dG(l')\right](1+r)\frac{dA}{dz}$$

But A need not be differentiable

if we can form this function $\Rightarrow \left.\frac{dw}{dz}\right|_{z^*} = \left.\frac{dV}{dz}\right|_{z^*}$

$$w(z) = \left\{ u(z - A(z^*)) + \beta \int V(wl + (1+r)A(z^*) - r\varphi)\, dG(l_{t+1}) \right\}$$

since we assume u is differentiable, w' exists and $w'(z^*) = u'(z^* - A(z^*))$

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This will give: $u'(z - A(z)) = \beta \int V'(\ \ )$

Result 2: If $c(z^*)=0 \Rightarrow c(z)=0 \;\forall\; z<z^*$

If $c(z^*)=0$, then it must be that $\varphi=\frac{w\ell_{min}}{r}$ and $z^*=z_{min}$

$$\Rightarrow V_+'(z) = \beta(1+r)\int V_+'(w\ell+(1+r)z-r\varphi)\,dG(\ell) \leq \underbrace{\beta(1+r)}_{<1 \text{ by assump.}} V_+'(z) < V_+'(z)$$

concavity of v since $w\ell+(1+r)z > w\ell+(1+r)z'$

which is a contradiction, thus proving $c(z)>0 \;\forall z>z^*$

It is possible that $v'(z)=u'(c) \geq \beta(1+r)E[v'(z')]$

with strict equality if φ is high enough.

Result3: $u'(0)$ finite or $\varphi > \frac{w\ell_{min}}{r}$, then $z_m = w\ell_{min} - r\varphi \geq 0$

Then $\exists\, \hat{z}>0$ s.t. $\forall z \leq \hat{z}$, $c=z \Rightarrow a'=0$

"Pf:" Contradiction.

$$v'(z) = \beta(1+r)E[v'(z')] \underset{\substack{\text{since } z' \geq z_{min} \\ \text{and } v \text{ concave}}}{\leq} \beta(1+r)v'(z_m) < v'(z_{min}) - \varepsilon$$

Take $z \to z_{min} \Rightarrow v'(z_{min}) < v'(z_{min}) - \varepsilon \;\; \rightarrow\!\leftarrow$

$u'(c) \quad < \quad u'(c+\varepsilon) \qquad E[\varepsilon]=0$

marginal utility increases when you throw in risk

$u'(c) \qquad (1+r)\beta u'(c+\varepsilon)$

Result4: $\forall z \geq z^*$, $\underset{\text{when } \ell=\ell_{max}}{y_{max}} + (1+r)A(z) \leq z$

If $\underset{\text{if bounded}}{A(z)} \leq K \Rightarrow \bar{R}K + y_{max} = z^*$ ✓

Suppose $A(z)$ is not bounded.

$v'(z) = u'(c(z)) \qquad z - A(z) = c(z)$

$$1 < \frac{E_t V'(z_{t+1})}{V'(y_{max}+RA(z_t))} \le \frac{V'(y_{min}+RA(z_t))}{V'(y_{max}+RA(z_t))} = \frac{u'(c(y_{min}+RA(z_t)))}{u'(c(y_{max}+RA(z_t)))} \quad (*)$$

Assume

$$\frac{-c\,u''(c)}{u'(c)} \le \mu \quad \forall c \ge \bar{c}$$

$$\Rightarrow u'(c) = Bc^{-\mu} \quad \text{where } B \text{ solves } u'(\bar{c}) = B\bar{c}^{-\mu}$$

$$\Rightarrow (*) = \left[\frac{c(y_{max}+RA(z))}{c(y_{min}+RA(z))}\right]^{\mu}$$

$$c(y_{max}+RA(z)) = c(y_{min}+RA(z)) + w(l_{max}-l_{min})h, \quad h \in (0,1)$$

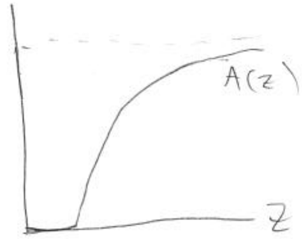
$$\Rightarrow (*) = \left[1 + \underbrace{\frac{w(l_{max}-l_{min})h}{c(y_{min}+RA(z))}}_{\to 0 \text{ as } z \to \infty}\right]^{\mu} \longrightarrow 1$$

Choose ε s.t. $\beta R(1+\varepsilon) < 1$

$$V'(z) = u'(c) = \beta R\, E[V'(z')] \le \beta R(1+\varepsilon)V'(y_{max}+RA(z_t))$$
$$< V'(y_{max}+RA(z_t))$$

$\Rightarrow y_{max}+RA(z_t) < z \Rightarrow$ decrease your asset position

A is capped.



$$z_{t+1} = w\underbrace{\ell_{t+1}}_{\text{stochastic}} + (1+r)A(z_t) - r\varphi$$

Can we characterize $f(z_{t+1}|z_t)$? Sure. Just solve for the ℓ. $\ell_{t+1} = \dfrac{z_{t+1} - (1+r)A(z_t) + r\varphi}{w}$

$$dG\left(\frac{z_{t+1} - (1+r)A(z_t) + r\varphi}{w}\right) = Q(z, z')$$

Simple first order Markov process since $\{\ell_t\}$ are iid