Knaster's Lemma

Let $f:[a,b]\to[a,b]$ be a function such that $f(x)>f(y)$ whenever $x>y$.
Prove that $\exists\ c\in[a,b]$ such that $f(c)=c$.

Hint: Assume by contradiction that there is no such c. Consider the set $A=\{x\in[a,b]\,|\,f(x)>x\}$ which is not empty (since $f(a)>a$) and bounded from above by b and hence has a supremum. Let $c=\sup A$. Show that neither $f(c)>c$ nor $f(c)<c$ is possible. (use monotonicity and the definition of $\sup$) and conclude that $f(c)=c$

Proof of Knaster's Lemma

Without loss of generality, assume $f(a)>a$. (If $f(a)=a$, we are done)
Define $A=\{x\in[a,b]\,|\,f(x)>x\}$. $a\in A$, so A is nonempty. $A\subset[a,b]$, which is a bounded set, so A is bounded. Therefore, by theorem 2.5.1, $\exists$ a least upper bound: $c=\sup A$.

Suppose $f(c)>c$. Let $d=c+\delta>c$. Then $f(d)>f(c)>c\Rightarrow d\in A$, which is a contradiction to $c=\sup A$. (Since $d>c$).

Suppose $f(c)<c$. Since $c=\sup A$, for all $x<c$, $f(x)>x$. In addition, we have that $f(x)<f(c)$ by monotonicity $\Rightarrow f(c)>x\ \ \forall\ x<c$.

Construct the sequence $x_n=c-\dfrac{(c-a)}{2^n}$. Since $x_n<c\ \forall n,$ we have that $\{x_n\}\subset A$. Thus, $f(x_n)>x_n\ \forall n$. In addition, by monotonicity, we have $f(x_n)<f(c)\ \forall n$. By transitivity, it follows that $f(c)>c-\dfrac{(c-a)}{2^n}\ \forall n$. Taking limits, we have that $f(c)\geq c$, which is a contradiction $\rightarrow\leftarrow$.

Therefore, it must be the case that $f(c)=c$. Q.E.D.