Section 9.1 The Heat Equation

Section 9.2 Definitions and Examples

Section 9.3 Pointwise Convergence

Lemma 9.3.1 (The Riemann-Lebesgue Lemma)

Let f be piecewise continuous on the finite interval $[a,b]$. Then,

$$\lim_{n\to\infty}\int_a^b f(t)e^{\pm int}\,dt = 0.$$

Theorem 9.3.2

Suppose that f is piecewise continuous on $[-\pi,\pi]$ and periodic of period 2π on $\mathbf{R}$. Let $\{c_m\}$ be the Fourier coefficients of f and define $S_n(x)=\sum_{m=-n}^{n} c_m e^{imx}$.

Then, at every point x where the left-hand and right-hand limits of the difference quotient of f exist,

$$\lim_{n\to\infty} S_n(x) = \frac{f(x^+)+f(x^-)}{2}.$$

Corollary 9.3.3

If f is 2π periodic and continuously differentiable, then the Fourier series of f converges to $f(x)$ at every x.

Section 9.4 Mean-square Convergence

Proposition 9.4.1 (The Cauchy-Schwarz Inequality)

For all piecewise continuous functions f and g,

$$|(f,g)| \le \|f\|_2\|g\|_2.$$

Proposition 9.4.2

If f and g are piecewise continuous on $[a,b]$, then

(a) $\|f\|_2 \ge 0$ and $\|f\|_2 = 0$ if and only if $f=0$.

(b) $\|\alpha f\|_2 = |\alpha|\|f\|_2$ for all $\alpha\in\mathbf{C}$.

(c) $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$.

Theorem 9.4.3

Let f be a piecewise continuous function and let $\{\varphi_n\}$ be an orthonormal family of piecewise continuous functions on a finite interval $[a,b]$. Then, choosing $c_n = (f, \varphi_n)$ minimizes $\left\| f - \sum_{n=1}^{N} c_n \varphi_n \right\|_2$, and the sequence $\{c_n\}$ satisfies Bessel's inequality.

Theorem 9.4.4

Let f be a continuously differentiable function of period 2π. Then, the Fourier series of f converges to f uniformly.

Lemma 9.4.5

Every periodic continuous function can be uniformly approximated by a periodic continuously differentiable function.

Theorem 9.4.6

Let f be a continuous function of period 2π and let $S_n(f) = \sum_{m=-n}^{n} c_m e^{-imx}$ be the partial sum of the Fourier series of f. Then,

$$\| f - S_n(f) \|_2 \to 0 \text{ as } n \to \infty$$

and

$$\sum_{m=-\infty}^{\infty} |c_m|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \, dx \qquad \text{(Parseval's relation)}$$