Section 6.1 Lim sup and Lim inf

Theorem 6.1.1

Let $\{a_n\}$ be a sequence of real numbers.

(a) If $\bar{s}$ is finite and $\varepsilon > 0$ is given, there exists an N so that $a_n \le \bar{s} + \varepsilon$ for all $n \ge N$, and for each N there exists an $n \ge N$ so that $a_n \ge \bar{s} - \varepsilon$. Conversely, if s is a number satisfying these properties, then $s = \bar{s}$

(b) If $\underline{s}$ is finite and $\varepsilon > 0$ is given, there exists an N so that $a_n \ge \underline{s} - \varepsilon$ for all $n \ge N$, and for each N there exists an $n \ge N$ so that $a_n \le \bar{s} + \varepsilon$. Conversely, if s is a number satisfying these properties, then $s = \underline{s}$.

Corollary 6.1.2

A sequence $\{a_n\}$ of real numbers converges to a finite limit a if and only if

$$-\infty < \limsup a_n \le \liminf < \infty$$

in which case

$$\limsup a_n = a = \liminf a_n.$$

Theorem 6.1.3

Let $\{a_n\}$ be a sequence of positive numbers. Then

$$\liminf \frac{a_{n+1}}{a_n} \le \liminf \sqrt[n]{a_n} \le \limsup \sqrt[n]{a_n} \le \limsup \frac{a_{n+1}}{a_n}.$$

Section 6.2 Series of Real Constants

Theorem 6.2.1

(a) A series $\sum_{j=1}^{\infty} a_j$ converges if and only if for each given $\varepsilon > 0$ there is an N such that

$$\left| \sum_{j=n}^{m} a_j \right| \le \varepsilon \quad \text{for all } n \ge N \text{ and } m \ge N.$$

(b) If $\sum a_j$ converges, then $a_j \to 0$ as $j \to \infty$.

(c) If $\sum |a_j|$ converges, then $\sum a_j$ converges.

(d) If $\sum a_j$ converges and $\sum b_j$ converges and c and d are any real numbers, then $\sum (ca_j + db_j)$ converges and

$$\sum (ca_j + db_j) = c\sum a_j + d\sum b_j.$$

Theorem 6.2.2 (The Comparison Test)

Let $\{a_j\}, \{b_j\}, \text{and} \{c_j\}$ be sequences of nonnegative numbers such that $a_j \le b_j \le c_j$ for each *j*. Then,

(a) If $\sum c_j$ converges, then $\sum b_j$ converges.

(b) If $\sum a_j$ diverges, then $\sum b_j$ diverges.

Theorem 6.2.3 (The Root Test)

Set $\alpha = \limsup |a_j|^{\frac{1}{j}}$. Then

(a) if $\alpha < 1$, the series $\sum a_j$ converges absolutely

(b) if $\alpha > 1$, the series $\sum a_j$ diverges.

Theorem 6.2.4 (The Ratio Test)

Let $\sum a_j$ be a series of nonzero terms. Then,

(a) if $\limsup \frac{|a_{j+1}|}{|a_j|} < 1$, the series converges absolutely.

(b) if $\liminf \frac{|a_{j+1}|}{|a_j|} > 1$, the series diverges.

Theorem 6.2.5

Suppose that $\sum a_j$ converges absolutely and let f be a one-to-one function from $\mathbf{N}$ to $\mathbf{N}$. Then, the series $\sum a_{f(j)}$ converges absolutely and $\sum a_{f(j)} = \sum a_j$.

Theorem 6.2.6

Suppose that $\sum a_j$ and $\sum b_k$ are absolutely convergent. Then $\sum a_j b_k$ is absolutely convergent and $\left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum a_j b_k$.

Section 6.3 The Weierstrass M-Test

Theorem 6.3.1 (The Weierstrass M-Test)

Let $\{f_j(x)\}$ be a sequence of functions defined on a set $E \subset \mathbf{R}$. Suppose that for each *j* there is a constant M_j such that $|f_j(x)| \le M_j$ for all $x \in E$ and that $\sum M_j$ converges. Then $S_n(x)$ converges uniformly to $f(x)$. If each f_j is continuous on E, then f is continuous on E.

Theorem 6.3.2

Let $\{f_j(x)\}$ be a sequence of continuous functions defined on a finite interval $[a,b]$. Suppose that the series $\sum_{j=1}^{\infty} f_j(x)$ converges uniformly to $f(x)$ on $[a,b]$. Then for each $x \in [a,b]$

$$\int_a^x f(t)dt = \sum_{j=1}^{\infty} \int_a^x f_j(t)dt,$$

and the series on the right converges uniformly in x.

Theorem 6.3.3

Let $\{f_j(x)\}$ be a sequence of continuously differentiable functions defined on a finite or infinite interval $[a,b]$. Suppose that the sum $\sum f_j(x)$ converges uniformly to $f(x)$ on $[a,b]$ and that $\sum f_j'(x)$ converges uniformly on $[a,b]$. Then, f is continuously differentiable on $[a,b]$ and

$$f'(x) = \sum_{j=1}^{\infty} f_j'(x).$$

Section 6.4 Power Series

Theorem 6.4.1

Let R be the radius of convergence of series $\sum_{j=0}^{\infty} a_j (x-x_0)^j$. Then the series converges for all x in the open interval $(x_0 - R, x_0 + R)$ and diverges for all x outside the closed interval $[x_0 - R, x_0 + R]$. For each $0 \le r < R$, the series converges uniformly on the interval $[x_0 - r, x_0 + r]$.

Theorem 6.4.2

Let R be the radius of convergence of the series $\sum_{j=0}^{\infty} a_j (x-x_0)^j$. Then the function $f(x)$ to which the series converges in $(x_0 - R, x_0 + R)$ is infinitely often continuously differentiable in $(x_0 - R, x_0 + R)$, and the derivatives can be computed by differentiating the series term by term. Furthermore, $\sum_{j=0}^{\infty} a_j (x-x_0)^j$ is just the Taylor series of f expanded about the point x_0.

Section 6.5 Complex Numbers

Theorem 6.5.1

A sequence of complex numbers converges if and only if it is a Cauchy sequence.

Theorem 6.5.2

Let $\{a_j\}, \{b_j\}$, and $\{c_j\}$ be sequences of complex numbers such that

$|a_j| \le |b_j| \le |c_j|$ for all j.

(a) If $\sum |c_j|$ converges, then $\sum b_j$ converges absolutely.

(b) If $\sum |a_j|$ diverges, then $\sum b_j$ does not converge absolutely.

Theorem 6.5.3

Let R be the radius of convergence of $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$. Then the series converges for all z in the open disk $\{z \mid |z - z_0| < R\}$ and diverges for all z outside the closed disk $\{z \mid |z - z_0| \le R\}$. The series converges uniformly on each closed subdisk $\{z \mid |z - z_0| \le r\}$ with $r < R$.