Section 5.1 Pointwise and Uniform Convergence

Section 5.2 Limit Theorems

Theorem 5.2.1

Let $\{f_n\}$ be a sequence of continuous functions on a set $E \subset \mathbf{R}$. If $f_n \to f$ uniformly on E as $n \to \infty$, then f is continuous on E.

Theorem 5.2.2

Let $\{f_n\}$ be a sequence of continuous functions on a finite interval $[a,b]$. If $f_n \to f$ uniformly as $n \to \infty$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Theorem 5.2.3

Let $\{f_n\}$ be a sequence of continuously differentiable functions on a finite interval $[a,b]$. Suppose that the sequence of derivatives $\{f_n'\}$ converges uniformly to a function g on $[a,b]$ and that for one point, x_0, the sequence of real numbers $\{f_n(x_0)\}$ converges. Then, $\{f_n\}$ converges uniformly to a continuous function f on $[a,b]$, f is continuously differentiable, and $f' = g$.

Theorem 5.2.4

Let f be a continuous function on the rectangle $Q \equiv [a,b] \times [c,d]$. Suppose that for each x the function $f(x,\cdot)$ is differentiable on $[c,d]$ and f_y is continuous on Q. Define a function F on $[c,d]$ by

$$F(y) \equiv \int_a^b f(x,y)dx.$$

Then F is continuously differentiable and

$$F'(y) = \int_a^b f_y(x,y)dx.$$

Section 5.3 The Supremum Norm

Proposition 5.3.1

Let f and g be functions defined on a set $E \subset \mathbf{R}$. Then,

(a) $\|f\|_\infty \ge 0$ and $\|f\|_\infty = 0$ if and only if f is the zero function on E.

(b) For every $\alpha \in \mathbf{R}$, we have $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty$.

(c) $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ (the triangle inequality)

Proposition 5.3.2

Let $\{f_n\}$ be a sequence of functions defined on a set $E \subset \mathbf{R}$. Let f be a function on E. Then, f_n converges to f uniformly on E if and only if f_n converges to f in the sup norm on E.

Theorem 5.3.3

Let $\{f_n\}$ be a sequence of continuous functions on $E \subset \mathbf{R}$. Suppose that $\{f_n\}$ is a Cauchy sequence in the sup norm. Then, there exists a continuous function f on E such that

$$\lim_{n\to\infty} \| f_n - f \|_\infty = 0.$$

Section 5.6 Metric Spaces

Proposition 5.6.1

Let ρ and σ be two equivalent metrics on a set $\mathbf{M}$ and suppose that $\{x_n\}$ is a sequence in $\mathbf{M}$. Then $x_n \to x$ in the metric ρ if and only if $x_n \to x$ in the metric σ.

Section 5.7 The Contraction Mapping Principle

Theorem 5.7.1 (The Contraction Mapping Principle)

Let T be a contraction on a complete metric space $(\mathbf{M}, \rho)$. Then there is a unique point $x \in \mathbf{M}$ such that $T(x) = x$. Furthermore, if x_0 is any point in $\mathbf{M}$ and we define $x_{n+1} = T(x_n)$, then $x_n \to x$ as $n \to \infty$.

Section 5.8 Normed Linear Spaces

Theorem 5.8.1

l_∞ is a Banach space.