Section 3.1 Continuity

Theorem 3.1.1

Let f and g be continuous functions and define $D = Dom(f) \bigcap Dom(g)$. Then,

(a) $f + g$ is continuous on D.

(b) For any constant κ, the function κf is continuous on $Dom(f)$.

(c) fg is continuous on D.

(d) $\dfrac{f}{g}$ is continuous at all $x \in D$ such that $g(x) \neq 0$.

Theorem 3.1.2

Let f and g be continuous functions and define

$$D = \{x \mid x \in Dom(g) \text{ and } g(x) \in Dom(f)\}$$

Then $f \circ g$ is continuous on D.

Theorem 3.1.3

A function $f(x)$ is continuous at a point c in $Dom(f)$ if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all x in $Dom(f)$:

$$|x - c| \leq \delta \qquad \text{implies} \; |f(x) - f(c)| \leq \varepsilon.$$

Section 3.2 Continuous Fcns on Closed Intervals

Theorem 3.2.1

A continuous function on a closed finite interval is bounded.

Theorem 3.2.2

Let f be a continuous function on a closed interval $[a,b]$. Then, there exist points c and d in $[a,b]$ such that

$$f(c) = \sup_{[a,b]} f \quad \text{and} \qquad f(d) = \inf_{[a,b]} f.$$

Theorem 3.2.3 (The Intermediate Value Theorem)

Let f be a continuous function on a closed interval [a,b] such that $f(a) \neq f(b)$. Let y be a real number between $f(a)$ and $f(b)$; that is, either $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there is a c in (a,b) such that $f(c) = y$.

Corollary 3.2.4

Let f be a continuous function on a closed interval $[a,b]$ and define $m \equiv \inf_{[a,b]} f$ and $M \equiv \sup_{[a,b]} f$. Then, the range fo f is the interval $[m, M]$.

Theorem 3.2.5

Let f be a continuous function on a closed finite interval $[a,b]$. Then, f is uniformly continuous on $[a,b]$.

Section 3.3 The Riemann Integral

Lemma 1

Let Q be a partition which contains the points of P and some additional points. Then $L_P(f) \le L_Q(f)$ and $U_Q(f) \le U_P(f)$

Lemma 2

Let P and Q be partitions. Then $L_P(f) \le U_Q(f)$.

Lemma 3

Let f be a bounded function on $[a,b]$. Suppose that for each $\varepsilon > 0$ there is a partition P so that

$$U_P(f) - L_P(f) \le \varepsilon.$$

Then f is Riemann integrable

Theorem 3.3.1

A continuous function on a closed interval is Riemann integrable.

Corollary 3.3.2

Suppose that f is a continuous function on $[a,b]$, and let $\{P_k\}$ be a sequence of partitions such that the maximum length of the subintervals goes to zero as $k \to \infty$. Let S_k be any Riemann sum corresponding to P_k. Then

$$S_k \to \int_a^b f(x)dx \text{ as } k \to \infty.$$

Theorem 3.3.3

Let f and g be continuous functions on the interval $[a,b]$ and let α and β be constants. Then

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.$$

Theorem 3.3.4

Let f and g be continuous functions on the interval $[a,b]$ and suppose that $f(x) \le g(x)$ for all $x \in [a,b]$. Then,

$$\int_a^b f(x)dx \le \int_a^b g(x)dx\,.$$

Theorem 3.3.5

Let f be a continuous function on the interval $[a,b]$. Then,

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Corollary 3.3.6

Let f be a continuous function on the interval $[a,b]$. Then,

$$\left|\int_a^b f(x)dx\right| \leq (b-a)\sup_{[a,b]}|f(x)|$$

Theorem 3.3.7

Let f be a continuous function on the interval $[a,b]$ and suppose that $a \leq c \leq b$. Then,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorems from Lecture

Theorem

Let $F : D \to \mathbf{R}$, $c \in \mathbf{R}$. Then $\lim_{x\to c} f(x) = L$ if and only if $\forall \varepsilon > 0\ \exists \delta > 0 \ni$ whenever $x \in D \setminus \{c\} \ni |x-c| < \delta$, $|f(x) - L| < \varepsilon$