

## Section 2.1 Convergence

## Section 2.2 Limit Theorems

### Proposition 2.2.1

Suppose that  $b_n \rightarrow b$ . Then the sequence  $\{b_n\}$  is bounded; that is, there is a number,  $M$ , so that

$$|b_n| \leq M, \text{ for all } n.$$

### Proposition 2.2.2 (The Squeeze Theorem)

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences such that  $a_n \rightarrow L$  and  $b_n \rightarrow L$ . Let  $\{c_n\}$  be a sequence such that  $a_n \leq c_n \leq b_n$  for all  $n$ . Then  $c_n \rightarrow L$ .

### Theorem 2.2.3

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences and suppose that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Then,

$$\lim_{n \rightarrow \infty} (a_n + b_n) = a + b.$$

### Theorem 2.2.4

Let  $\{a_n\}$  be a sequence and suppose that  $a_n \rightarrow a$ . Then, for any constant  $\kappa$ ,

$$\lim_{n \rightarrow \infty} (\kappa a_n) = \kappa \lim_{n \rightarrow \infty} (a_n) = \kappa a.$$

### Theorem 2.2.5

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences and suppose that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Then,

$$\lim_{n \rightarrow \infty} (a_n b_n) = ab.$$

### Theorem 2.2.6

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences and suppose that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Suppose that  $b \neq 0$  and  $b_n \neq 0$  for any  $n$ . Then,

$$\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \frac{a}{b}.$$

## Section 2.4 Cauchy Sequences

### Proposition 2.4.1

If  $\{a_n\}$  converges to a finite limit, then  $\{a_n\}$  is a Cauchy sequence.

## **Axiom of Completeness**

Every Cauchy sequence of real numbers converges to a finite real number.

### **Theorem 2.4.2**

A sequence of real numbers is a Cauchy sequence if and only if it converges to a finite real number.

### **Theorem 2.4.3**

Every bounded monotone sequence converges.

## **Section 2.5 Supremum and Infimum**

### **Theorem 2.5.1**

Let  $S$  be a set of real numbers which is bounded above. Then  $S$  has a unique least upper bound.

### **Theorem 2.5.2**

Let  $a$  and  $b$  be real numbers which satisfy  $0 < a < b$ . Then there is a positive integer  $n$  such that  $b < na$ .

### **Theorem 2.5.3**

Let  $A$  and  $B$  be two sets of real numbers and suppose that for each  $a \in A$  and  $b \in B$  the inequality  $a \leq b$  holds. Then  $\sup A \leq \inf B$ .

## **Section 2.6 The Bolzano-Weierstrass Theorem**

### **Proposition 2.6.1**

Let  $\{a_n\}$  be a sequence of real numbers. Then  $d$  is a limit point of the sequence if and only if there exists a subsequence  $\{a_{n_k}\}$  of  $\{a_n\}$  which converges to  $d$ .

### **Theorem 2.6.2 (The Bolzano-Weierstrass Theorem)**

Every bounded sequence has a convergent subsequence.

### **Corollary 2.6.3**

If  $\{a_n\}$  is a sequence of numbers in the closed interval  $[b, c]$ , then a subsequence of  $\{a_n\}$  converges to a point in  $[b, c]$ .