Michael Powell; 603-210-926
Econ 201B; Prof. Levine
University of California at Los Angeles
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Problem Set 2

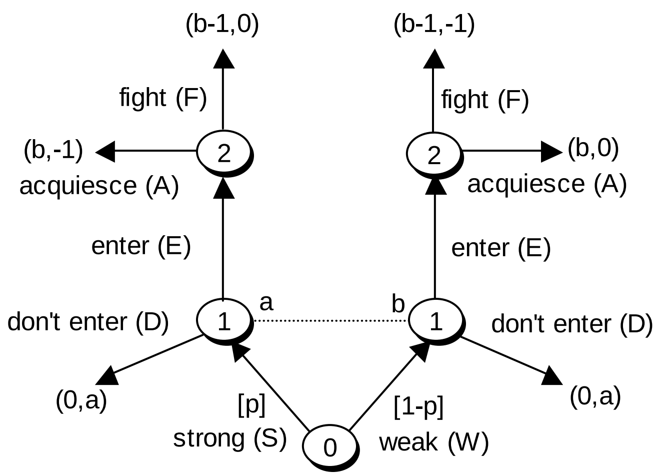
**The Chain Store Paradox Paradox**

Consider the Kreps-Wilson version of the chain store paradox: An entrant may stay out and get nothing $(0)$, or he may enter. If he enters, the incumbent may fight or acquiesce. The entrant gets $b$ if the incumbent acquiesces, and $b-1$ if he fights, where $0<b<1$. There are two types of incumbent, both receiving $a>1$ if there is no entry. If there is a fight, the strong incumbent gets 0 and the weak incumbent gets –1; if a strong incumbent acquiesces he gets –1, a weak incumbent 0.

Only the incumbent knows whether he is weak or strong; it is common knowledge that the entrant a priori believes that he has a $p$ chance of facing a strong incumbent. Define

$$\gamma = \frac{p}{1-p}\frac{1-b}{b}$$

1.a Sketch the extensive form of this game:




1.b Define a sequential equilibrium of this game.

There are two preliminary definitions that are important for understanding what a sequential equilibrium is: (See Campos, 2005)

A belief $b_i = (\alpha_i, \mu_i)$ is consistent if $\alpha_i = \lim \alpha_i^n$, where $\alpha_i^n$ is the assessment that was constructed by using Bayes rule on a strictly positive sequence $\mu_i^n \to \mu_i$.

A behavior strategy $\pi_i$ is <u>sequentially rational</u> for player $i$ if there is no profitable deviation $\pi_i' \in \Pi_i$, given beliefs $b_i$ for player $i$, i.e. $u_i(\pi_i \mid b_i) \geq u_i(\pi_i' \mid b_i), \forall \pi_i' \in \Pi_i$.

A <u>sequential equilibrium</u> is a behavior strategy profile $\pi$ and an assessment $\alpha_i$ for each player $i \in \{1,2\}$ such that $(\alpha_i, \pi_{-i})$ is consistent and $\pi_i$ is sequentially rational for each $i \in \{1,2\}$.

1.c Show that if $\gamma \neq 1$, there is a unique sequential equilibrium, and that if $\gamma > 1$ entry never occurs, while if $\gamma < 1$ entry always occurs.

In all these cases, the assessment for the entrant is correct, since node *a* is reached with exactly probability *p* regardless of what the incumbent does and node *b* is reached with exactly probability $1-p$ regardless of what the incumbent does. Therefore, we have: $(\alpha_1(a), \alpha_1(b)) = (p, 1-p)$. Since these assessments are constructed using Bayes rule, we have that the only beliefs that are consistent are those with $(\alpha_1(a), \alpha_1(b)) = (p, 1-p)$.

Are there any pooling equilibria?

$(A, A)$:
Consider the case where both the "strong" and the "weak" incumbents acquiesce. That is, where $\pi_2(S) = \pi_2(W) = A$.

If both incumbents acquiesce, it is a best response for the entrant to enter:

$$\left.\begin{array}{l} u_1(\pi_1, \pi_2(S), \pi_2(W)) = u_1(E, A, A) = (p)(b) + (1-p)(b) = b \\ u_1(\pi_1, \pi_2(S), \pi_2(W)) = u_1(D, A, A) = (p)(0) + (1-p)(0) = 0 \end{array}\right\} \; b > 0 \Rightarrow \text{Enter}$$

Given that the entrant is going to enter, however, it is not a best response for the "weak" incumbent to fight:

$$\left.\begin{array}{l} u_2(\pi_2(\theta_2), \pi_1 \mid \theta_2 = W) = u_2(F, E \mid W) = -1 \\ u_2(\pi_2'(\theta_2), \pi_1 \mid \theta_2 = W) = u_2(A, E \mid W) = 0 \end{array}\right\} \; 0 > -1 \Rightarrow \text{Acquiesce}$$

Therefore, this is not a sequential equilibrium because it is not sequentially rational for player 2.

$(F, F)$:
Consider the case where both the "strong" and the "weak" incumbents fight. That is, where $\pi_2(S) = \pi_2(W) = F$.

If both incumbents fight, it is a best response for the entrant not to enter:

$$\left.\begin{array}{l} u_1(\pi_1, \pi_2(S), \pi_2(W)) = u_1(D, F, F) = (p)(0) + (1-p)(0) = 0 \\ u_1(\pi_1, \pi_2(S), \pi_2(W)) = u_1(E, F, F) = (p)(b-1) + (1-p)(b-1) = b-1 \end{array}\right\}$$

Since $b-1<0,$ it is a best response not to enter for player 1.

Given that the entrant is not going to enter, however, there are no beliefs for the weak incumbent that would support him fighting. If the weak incumbent is ever allowed to move, it is a strictly dominant strategy to acquiesce. Therefore, this is not a sequential equilibrium because it is not sequentially rational for player 2.

Are there any separating equilibria?

$(A,F):$
Consider the case where the "strong" incumbent acquiesces and the "weak" incumbent fights. That is, where $\pi_2(S)=A$ and $\pi_2(W)=F.$

Following similar logic to the $(F,F)$ case, if the weak incumbent is ever allowed to move, it is a strictly dominant strategy to acquiesce. Similarly, if the strong incumbent is ever allowed to move, it is a strictly dominant strategy to fight. Therefore, this is not a sequential equilibrium, because it is not sequentially rational for player 2.

In addition, the same logic can be applied to any convex combinations of the above cases. Therefore, any quasi-separating equilibria involve placing positive probability on a strictly dominated strategy for player II, which cannot be an equilibrium. Thus, the only case that remains is:

$(F,A):$
Consider the case where the "strong" incumbent fights and the "weak" incumbent acquiesces. That is, where $\pi_2(S)=F$ and $\pi_2(W)=A.$

$$u_1(\pi_1,\pi_2(S),\pi_2(W))=u_1(E,F,A)=(p)(b-1)+(1-p)(b)$$
$$u_1(\pi_1,\pi_2(S),\pi_2(W))=u_1(D,F,A)=(p)(0)+(1-p)(0)=0$$

The best response for the entrant is therefore determined by the relative magnitudes of these payoffs:

If $(p)(b-1)+(1-p)(b)>0,$ then it is a best response for the entrant to enter. This occurs when $\gamma=\frac{p}{1-p}\frac{1-b}{b}<1.$ That is, whenever $\gamma<1,$ we have a unique sequential equilibrium where the entrant enters, the "strong" incumbent fights, and the "weak" incumbent acquiesces.

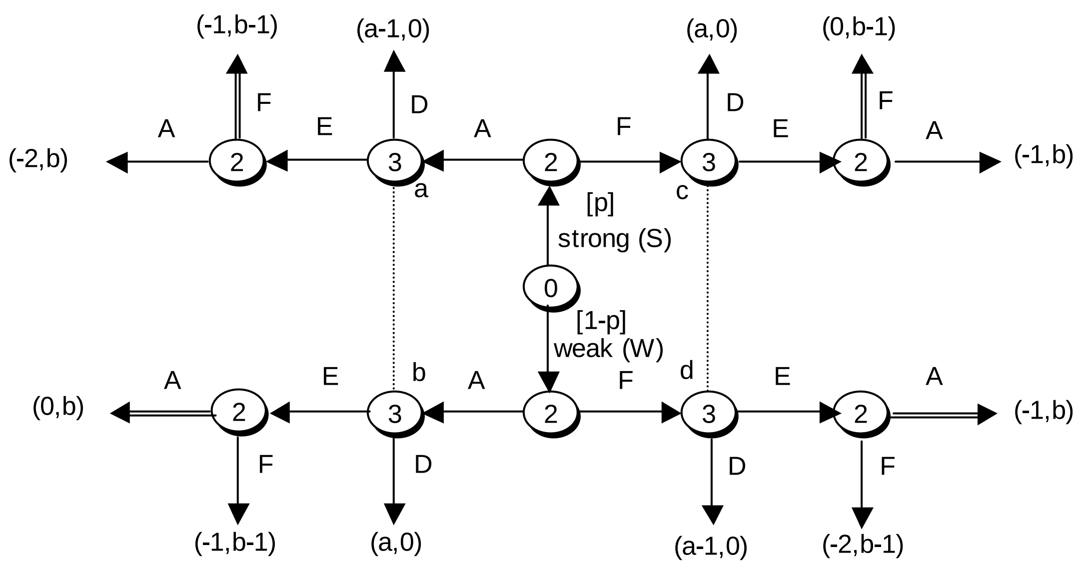
If $(p)(b-1)+(1-p)(b)<0,$ then it is a best response for the entrant not to enter. This occurs when $\gamma=\frac{p}{1-p}\frac{1-b}{b}>1.$ That is, whenever $\gamma>1,$ we have a unique sequential equilibrium where the entrant does not enter, the "strong" incumbent fights, and the "weak" incumbent acquiesces.

1.d What are the sequential equilibria if $\gamma = 1$?

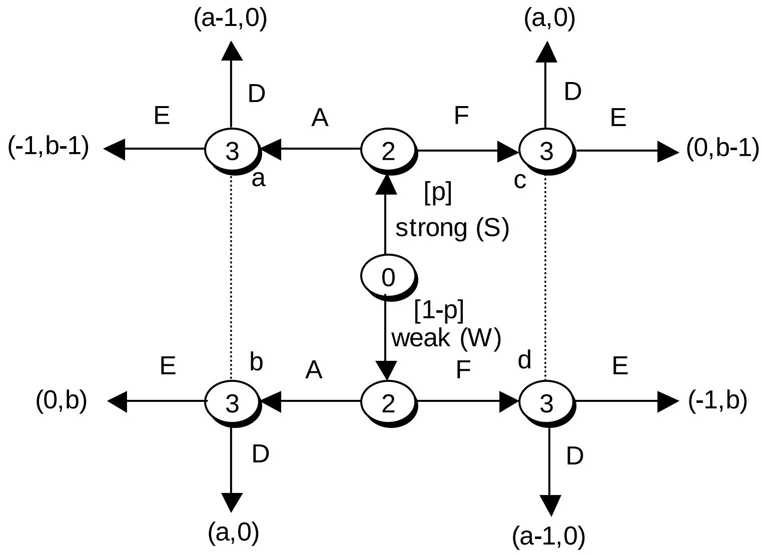
When $\gamma = 1$, we have that $(p)(b-1)+(1-p)(b)=0$, that is: $u_1(E,F,A)=u_1(D,F,A)$. In such a case, the entrant is indifferent between entering and not entering. Therefore, both behavior strategies by the entrant are acceptable as is any mixing between the two. Therefore, any behavior strategy profiles of the form:
$\pi = (\pi_1, \pi_2(S), \pi_2(W)) = (p, 1-p, F, A)$ is sequentially rational and beliefs are correct. Therefore, all such $\pi$ satisfy the criteria for a sequential equilibrium.

1.e Now suppose that the incumbent plays a second round against a different entrant who knows the result of the first round. The incumbent's goal is to maximize the sum of his payoffs in the two rounds. Show that if $\gamma > 1$ there is a sequential equilibrium in which the entrant enters on the first round and both types of incumbents acquiesce. Be careful to specify both the equilibrium strategies and beliefs.

With $\gamma > 1$, given that both types of incumbents will acquiesce, it is perfectly rational for the first entrant to enter. The second round game simply becomes a signaling game. Clearly in the final nodes, there are dominant strategies for the incumbent firms. I have indicated them with double lines. The payoffs are of the form: $(u_2, u_3)$.




Applying backward induction, this game becomes a game which is almost identical in form to the Beer-Quiche game:




The question then becomes identical to: "Is there a pooling equilibrium in this game in which both types of incumbent acquiesce?"

$(A, A)$:

The assessments for the entrant for the left information set using Bayes law are:

$$\alpha_3(a) = \frac{p}{p+1-p} = p; \ \ \alpha_3(b) = \frac{1-p}{p+1-p} = 1-p$$

$$u_3(E \mid A) = p(b-1) + (1-p)b < 0 \ \text{ since } \ \gamma > 1$$

$$u_3(D \mid A) = 0$$

$\Rightarrow$ The entrant will not enter.

Are there possible assessments for the entrant for the right information set which would cause him to enter (and thus would not give any profitable incentive for either type of incumbent to deviate)?

Consider the sequence of behavior strategies $(\pi_2(A \mid S), \pi_2(F \mid S), \pi_2(A \mid W), \pi_2(F \mid W))$
$= (1-\varepsilon_1, \varepsilon_1, 1-\varepsilon_2, \varepsilon_2)$

Then the assessments for the entrant for the right information set are:

$$\alpha_3(c) = \frac{\varepsilon_1 p}{\varepsilon_1 p + \varepsilon_2(1-p)}; \ \ \alpha_3(d) = \frac{\varepsilon_2(1-p)}{\varepsilon_1 p + \varepsilon_2(1-p)}$$

$$u_3(E \mid F) = \frac{\varepsilon_1 p}{\varepsilon_1 p + \varepsilon_2(1-p)}(b-1) + \frac{\varepsilon_2(1-p)}{\varepsilon_1 p + \varepsilon_2(1-p)} b$$

$$u_3(D \mid F) = 0$$

The question then becomes: are there any values of $\varepsilon_1$ and $\varepsilon_2$ which would cause the incumbent to enter in this case?

$$\frac{\varepsilon_1 p(b-1)}{\varepsilon_1 p+\varepsilon_2(1-p)}+\frac{\varepsilon_2(1-p)b}{\varepsilon_1 p+\varepsilon_2(1-p)}>0 \Leftrightarrow \varepsilon_1 p(b-1)+\varepsilon_2(1-p)b>0$$

$\Leftrightarrow \varepsilon_2(1-p)b>\varepsilon_1 p(1-b) \Leftrightarrow \frac{\varepsilon_2}{\varepsilon_1}>\frac{p}{1-p}\frac{1-b}{b}=\gamma$. Thus, as long as we can construct a sequence of behavior strategies $(\pi_2(A|S),\pi_2(F|S),\pi_2(A|W),\pi_2(F|W))$ $=(1-\varepsilon_1,\varepsilon_1,1-\varepsilon_2,\varepsilon_2)$ for the incumbent firm where $\varepsilon_2>\gamma\varepsilon_1$, the beliefs which cause the second entrant not to enter are consistent. Therefore, this is a sequential equilibrium.

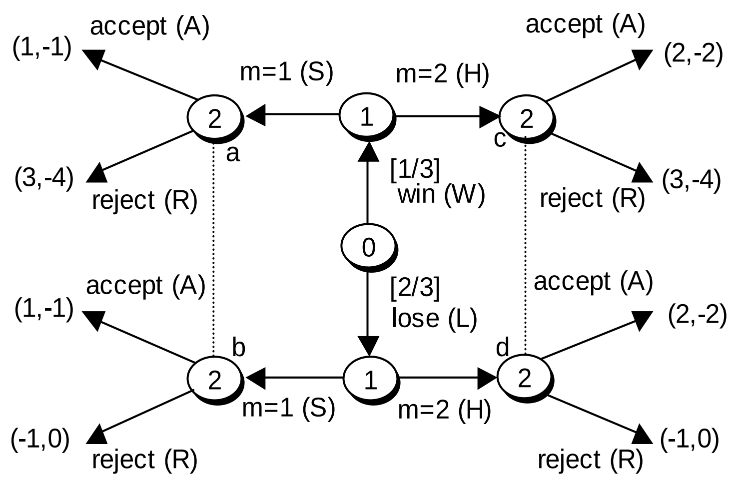
**Courtroom Drama**

Two players: plaintiff and defendant, in a civil suit. The plaintiff knows whether or not he will win the case if it goes to trial, but the defendant does not. The defendant's beliefs are Pr(plaintiff wins)=$\frac{1}{3}$. This is common knowledge

The cost of the trial is 1. The loser of the trial bears this cost. If the plaintiff wins the trial, then the defendant will have to pay the plaintiff 3 and also pay for the cost of the trial. If the plaintiff loses, he'll have to pay for the cost of the trial, and the defendant will neither win anything nor lose anything.

The plaintiff has two actions: ask for a low settlement, $m = 1$, or ask for a high settlement, $m = 2$. If the defendant accepts *m*, then the defendant is agreeing to pay *m* to the plaintiff out of court. If the defendant rejects *m*, the case goes to court.

2.a Draw the game tree.




2.b Find all sequential equilibria.

Are there any separating sequential equilbria?

$(H, S)$:

Consider the case where the "winning" plaintiff asks for a high settlement and the "losing" plaintiff asks for a small settlement. That is, where $\pi_1(W) = H$ and $\pi_1(L) = S$.

Since both information sets for the defendant are reached with positive probability, we can create assessments for each information set using Bayes law:

$$\alpha_2(a) = \frac{(0)\left(\frac{1}{3}\right)}{(0)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right)} = 0; \quad \alpha_2(b) = \frac{(1)\left(\frac{2}{3}\right)}{(0)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right)} = 1$$

$$\alpha_2(c) = \frac{(1)\left(\frac{1}{3}\right)}{(1)\left(\frac{1}{3}\right) + (0)\left(\frac{2}{3}\right)} = 1; \quad \alpha_2(d) = \frac{(0)\left(\frac{2}{3}\right)}{(1)\left(\frac{1}{3}\right) + (0)\left(\frac{2}{3}\right)} = 0.$$

Under such assessments, we have:

$$\left.\begin{array}{l} u_2(A \mid H) = (1)(-2) + (0)(-2) = -2 \\ u_2(R \mid H) = (1)(-4) + (0)(0) = -4 \end{array}\right\} \text{The defendant will accept a high settlement.}$$

$$\left.\begin{array}{l} u_2(A \mid S) = (0)(-1) + (1)(-1) = -1 \\ u_2(R \mid S) = (0)(-4) + (1)(0) = 0 \end{array}\right\} \text{The defendant will reject a small settlement.}$$

Given that the defendant will accept any high offer and reject any small offer, there exists a profitable deviation for the plaintiff:

$$u_1(\pi_1(\theta), \pi_2(H), \pi_2(S) \mid \theta = L) = u_1(S, A, R \mid L) = -1$$
$$u_1(\pi_1'(\theta), \pi_2(H), \pi_2(S) \mid \theta = H) = u_1(H, A, R \mid L) = 2$$

Therefore, a losing plaintiff will have the incentive to deviate to asking for a high settlement. i.e. this behavior strategy is not sequentially rational for the plaintiff.

$(S, H)$:

Consider the case where the "winning" plaintiff asks for a small settlement and the "losing" plaintiff asks for a high settlement. That is, where $\pi_1(W) = S$ and $\pi_1(L) = H$.

Since both information sets for the defendant are reached with positive probability, we can create assessments for each information set using Bayes law:

$$\alpha_2(a) = \frac{(1)\left(\frac{1}{3}\right)}{(1)\left(\frac{1}{3}\right) + (0)\left(\frac{2}{3}\right)} = 1; \quad \alpha_2(b) = \frac{(0)\left(\frac{2}{3}\right)}{(1)\left(\frac{1}{3}\right) + (0)\left(\frac{2}{3}\right)} = 0$$

$$\alpha_2(c) = \frac{(0)\left(\frac{1}{3}\right)}{(0)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right)} = 0; \quad \alpha_2(d) = \frac{(1)\left(\frac{2}{3}\right)}{(0)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right)} = 1.$$

Under such assessments, we have:

$$\left.\begin{array}{l} u_2(A \mid H) = (0)(-2) + (1)(-2) = -2 \\ u_2(R \mid H) = (0)(-4) + (1)(0) = 0 \end{array}\right\} \text{The defendant will reject a high settlement.}$$

$$\left.\begin{array}{l} u_2(A \mid S) = (1)(-1) + (0)(-1) = -1 \\ u_2(R \mid S) = (1)(-4) + (0)(0) = -4 \end{array}\right\} \text{The defendant will accept a small settlement.}$$

Given that the defendant will accept any small offer and reject any high offer, there exists a profitable deviation for the plaintiff:

$$u_1(\pi_1(\theta),\pi_2(H),\pi_2(S)\,|\,\theta=W)=u_1(S,R,A|W)=1$$

$$u_1(\pi_1'(\theta),\pi_2(H),\pi_2(S)\,|\,\theta=W)=u_1(H,R,A|W)=3$$

Therefore, a winning plaintiff will have the incentive to deviate to asking for a high settlement. i.e. this behavior strategy is not sequentially rational for the plaintiff.

Are there any pooling equilibria?

$(S,S)$:

Consider the case where the "winning" plaintiff asks for a small settlement and the "losing" plaintiff asks for a small settlement. That is, where $\pi_1(W)=\pi_2(L)=S$.

Since the information set on the left will be reached with positive probability, by Bayes law, we have the following assessments for the defendant:

$$\alpha_2(a)=\frac{(1)\left(\frac{1}{3}\right)}{(1)\left(\frac{1}{3}\right)+(1)\left(\frac{2}{3}\right)}=\frac{1}{3};\;\; \alpha_2(b)=\frac{(1)\left(\frac{2}{3}\right)}{(1)\left(\frac{1}{3}\right)+(1)\left(\frac{2}{3}\right)}=\frac{2}{3}$$

Under such assessments, we have:

$$\left.\begin{aligned} u_2(A\,|\,S)&=\left(\frac{1}{3}\right)(-1)+\left(\frac{2}{3}\right)(-1)=-1\\ u_2(R\,|\,S)&=\left(\frac{1}{3}\right)(-4)+\left(\frac{2}{3}\right)(0)=-\frac{4}{3}\end{aligned}\right\}\text{ The defendant will accept a small settlement.}$$

Given that the defendant will accept any small settlement, are there any profitable deviations for the plaintiff (i.e. is this sequentially rational for the plaintiff?)

Under these conditions, the "winning" plaintiff will receive 1 by asking for a small settlement since the defendant will accept.

Construct the general sequence of beliefs for the defendant:

$$\mu_2(\pi_1(S\,|W),\pi_1(H\,|W),\pi_1(S\,|\,L),\pi_1(H\,|\,L))=(1-\varepsilon_1,\varepsilon_1,1-\varepsilon_2,\varepsilon_2),\;\; \varepsilon_1\in(0,1),\; \varepsilon_2\in(0,1)$$

The assessments over *c* and *d* which are consistent with these beliefs are:

$$\alpha_2(c)=\frac{\varepsilon_1\frac{1}{3}}{\varepsilon_1\frac{1}{3}+\varepsilon_2\frac{2}{3}}=\frac{\varepsilon_1}{\varepsilon_1+2\varepsilon_2};\;\; \alpha_2(d)=\frac{\varepsilon_2\frac{2}{3}}{\varepsilon_1\frac{1}{3}+\varepsilon_2\frac{2}{3}}=\frac{2\varepsilon_2}{\varepsilon_1+2\varepsilon_2}.$$

However, under any such assessments, we have:

$$u_1(\pi_1(\theta),\pi_2(H),\pi_2(S)\,|\,\theta=W)=u_1(S,R,A|W)=1$$
$$u_1(\pi_1'(\theta),\pi_2(H),\pi_2(S)\,|\,\theta=W)=u_1(H,R,A|W)\geq \min\{2,3\}=2$$

Therefore, a winning plaintiff will have the incentive to deviate to asking for a high settlement. i.e. this behavior strategy is not sequentially rational for the plaintiff.

$(S,S)$:
Consider the case where the "winning" plaintiff asks for a small settlement and the "losing" plaintiff asks for a small settlement. That is, where $\pi_1(W)=\pi_2(L)=S$.

Since the information set on the left will be reached with positive probability, by Bayes law, we have the following assessments for the defendant:

$$\alpha_2(a)=\frac{(1)\left(\frac{1}{3}\right)}{(1)\left(\frac{1}{3}\right)+(1)\left(\frac{2}{3}\right)}=\frac{1}{3};\;\; \alpha_2(b)=\frac{(1)\left(\frac{2}{3}\right)}{(1)\left(\frac{1}{3}\right)+(1)\left(\frac{2}{3}\right)}=\frac{2}{3}$$

Under such assessments, we have:

$$\left.\begin{array}{l} u_2(A|S)=\left(\frac{1}{3}\right)(-1)+\left(\frac{2}{3}\right)(-1)=-1 \\ u_2(R|S)=\left(\frac{1}{3}\right)(-4)+\left(\frac{2}{3}\right)(0)=-\frac{4}{3} \end{array}\right\}$$ The defendant will accept a small settlement.

Given that the defendant will accept any small settlement, are there any profitable deviations for the plaintiff (i.e. is this sequentially rational for the plaintiff?)

Under these conditions, the "winning" plaintiff will receive 1 by asking for a small settlement since the defendant will accept.

Construct the general sequence of beliefs for the defendant:
$\mu_2(\pi_1(S\,|W),\pi_1(H\,|W),\pi_1(S\,|\,L),\pi_1(H\,|\,L))=(1-\varepsilon_1,\varepsilon_1,1-\varepsilon_2,\varepsilon_2)$, $\varepsilon_1\in(0,1)$, $\varepsilon_2\in(0,1)$
The assessments over *c* and *d* which are consistent with these beliefs are:

$$\alpha_2(c)=\frac{\varepsilon_1\frac{1}{3}}{\varepsilon_1\frac{1}{3}+\varepsilon_2\frac{2}{3}}=\frac{\varepsilon_1}{\varepsilon_1+2\varepsilon_2};\;\; \alpha_2(d)=\frac{\varepsilon_2\frac{2}{3}}{\varepsilon_1\frac{1}{3}+\varepsilon_2\frac{2}{3}}=\frac{2\varepsilon_2}{\varepsilon_1+2\varepsilon_2}.$$

However, under any such assessments, we have:

$$u_1(\pi_1(\theta),\pi_2(H),\pi_2(S)\,|\,\theta=W)=u_1(S,R,A|W)=1$$
$$u_1(\pi_1'(\theta),\pi_2(H),\pi_2(S)\,|\,\theta=W)=u_1(H,R,A|W)\geq \min\{2,3\}=2$$

Therefore, a winning plaintiff will have the incentive to deviate to asking for a high settlement. i.e. this behavior strategy is not sequentially rational for the plaintiff.

$(H, H)$:

Consider the case where the "winning" plaintiff asks for a high settlement and the "losing" plaintiff asks for a high settlement. That is, where $\pi_1(W) = \pi_2(L) = H$.

Since the information set on the right will be reached with positive probability, by Bayes law, we have the following assessments for the defendant:

$$\alpha_2(c) = \frac{(1)\left(\frac{1}{3}\right)}{(1)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right)} = \frac{1}{3}; \;\; \alpha_2(d) = \frac{(1)\left(\frac{2}{3}\right)}{(1)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right)} = \frac{2}{3}$$

Under such assessments, we have:

$$\left.\begin{aligned} u_2(A \mid H) &= \left(\frac{1}{3}\right)(-2) + \left(\frac{2}{3}\right)(-2) = -2 \\ u_2(R \mid H) &= \left(\frac{1}{3}\right)(-4) + \left(\frac{2}{3}\right)(0) = -\frac{4}{3} \end{aligned}\right\}$$ The defendant will reject a high settlement.

Given that the defendant will reject any high settlement, is this behavior strategy sequentially rational for the plaintiff?

Construct the sequence of beliefs for the defendant which converge to the behavior strategy of the plaintiff:

$\mu_2(\pi_1(S \mid W), \pi_1(H \mid W), \pi_1(S \mid L), \pi_1(H \mid L)) = (\varepsilon_1, 1-\varepsilon_1, \varepsilon_2, 1-\varepsilon_2)$, $\varepsilon_1 \in (0,1)$, $\varepsilon_2 \in (0,1)$

The assessments over *a* and *b* which are consistent with these beliefs are:

$$\alpha_2(a) = \frac{\varepsilon_1 \frac{1}{3}}{\varepsilon_1 \frac{1}{3} + \varepsilon_2 \frac{2}{3}} = \frac{\varepsilon_1}{\varepsilon_1 + 2\varepsilon_2}; \;\; \alpha_2(b) = \frac{\varepsilon_2 \frac{2}{3}}{\varepsilon_1 \frac{1}{3} + \varepsilon_2 \frac{2}{3}} = \frac{2\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}.$$

$$u_2(A \mid S) = \left(\frac{\varepsilon_1}{\varepsilon_1 + 2\varepsilon_2}\right)(-1) + \left(\frac{2\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right)(-1) = -1$$

$$u_2(R \mid S) = \left(\frac{\varepsilon_1}{\varepsilon_1 + 2\varepsilon_2}\right)(-4) + \left(\frac{2\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right)(0) = -\frac{4\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}$$

If we can find a sequence under which the defendant will reject all small settlements, then neither the "winning" plaintiff nor the "losing" plaintiff will have any profitable incentives to deviate. (The "winning" plaintiff will strictly prefer not to deviate and the "losing" plaintiff will be indifferent between deviating and not deviating)

Under what circumstances will the defendant reject all small settlements? i.e. when is:

$$\frac{-4\varepsilon_1}{\varepsilon_1+2\varepsilon_2} > -1?$$

$$\frac{-4\varepsilon_1}{\varepsilon_1+2\varepsilon_2} > -1 \Leftrightarrow \frac{4\varepsilon_1}{\varepsilon_1+2\varepsilon_2} < 1 \Leftrightarrow 4\varepsilon_1 < \varepsilon_1 + 2\varepsilon_2 \Leftrightarrow 3\varepsilon_1 < 2\varepsilon_2 \Leftrightarrow \varepsilon_1 < \frac{2}{3}\varepsilon_2$$

Therefore, we can construct a sequence of believes for the defendant which converge to the plaintiff's behavior strategy and allow the plaintiff's behavior strategy to be sequentially rational:

$\mu_2(\pi_1(S|W),\pi_1(H|W),\pi_1(S|L),\pi_1(H|L)) = (\varepsilon_1, 1-\varepsilon_1, \varepsilon_2, 1-\varepsilon_2)$ where $\varepsilon_1 \in (0,1)$ and $\varepsilon_1 < \frac{2}{3}\varepsilon_2$.

That is, this behavior strategy profile is a sequential equilibrium.

Are there any quasi-separating sequential equilibria?

I will only briefly discuss this idea, since the detail necessary to prove the existence or lack of existence of these is long and tedious (and I'm sure you don't want to read it any more than I want to write it).

All behavior strategies for the plaintiff discussed in this section will be of the form: $(\pi_1(S|W),\pi_1(H|W),\pi_1(S|L),\pi_1(H,L))$. Any variables *p* used will be assumed to be such that $p \in (0,1)$.

Consider the behavior strategies of the form: $(1,0,p,1-p)$

Under such behavior strategies, the defendant knows that if he is offered a high settlement, that it is the "losing" plaintiff making the offer. Therefore, he will reject it. The "winning" plaintiff, knowing this, has profitable incentive to deviate to a high settlement for the reasons given in the discussion of the $(S,S)$ pooling behavior strategy. Therefore, all behavior strategies of the form: $(1,0,p,1-p)$ are not sequential equilibria.

Consider the behavior strategies of the form: $(0,1,p,1-p)$, $p \in \left(\frac{1}{2},1\right)$

Under such behavior strategies, the defendant knows that if he is offered a low settlement, that it is the "losing" plaintiff making the offer. Therefore, he will reject it. But he will also reject any high settlements (as in the case of the $(H,H)$ pooling behavior strategy). As such, neither the "winning" plaintiff nor the "losing" plaintiff will have any profitable incentive to deviate and offer a small settlement (they will both be indifferent between offering a high settlement and a low settlement). Therefore, all behavior strategies of the form: $(0,1,p,1-p)$, $p \in \left(\frac{1}{2},1\right)$ are sequential equilibria.

Consider the behavior strategies of the form: $(0,1,p,1-p)$, $p \in \left(0,\frac{1}{2}\right)$

Under such behavior strategies, the defendant knows that if he is offered a low settlement, that it is the "losing" plaintiff making the offer. Therefore, he will reject it. He will also accept any high settlements. Given this behavior strategy for the defendant, the "losing" plaintiff will have the profitable incentive to decrease the probability that he offers a small settlement. That is, $\forall p \in \left(0,\frac{1}{2}\right)$, $\exists p' \in \left(0,\frac{1}{2}\right)$ with $p' < p$ such that the plaintiff would strictly prefer to use the behavior strategy $(0,1,p',1-p')$ to $(0,1,p,1-p)$. That is, there is a profitable deviation. i.e. All behavior strategies of the form: $(0,1,p,1-p)$, $p \in \left(0,\frac{1}{2}\right)$ are not sequential equilibria.

Consider the behavior strategies of the form: $(p,1-p,1,0)$, $p \in \left(\frac{2}{3},1\right)$.

Under such behavior strategies, the defendant knows that if he is offered a high settlement, that it is the "winning" plaintiff making the offer. Therefore, he will accept it. He will also accept any low settlement. Given this behavior strategy for the defendant, the "winning" plaintiff will have the profitable incentive to increase the probability that he offers a high settlement. That is, $\forall p \in \left(\frac{2}{3},1\right)$, $\exists p' \in \left(\frac{2}{3},1\right)$ with $p' < p$ such that the plaintiff would strictly prefer to use the behavior strategy $(p',1-p',1,0)$ to $(p,1-p,1,0)$. That is, there is a profitable deviation. i.e. All behavior strategies of the form: $(p,1-p,1,0)$, $p \in \left(\frac{2}{3},1\right)$ are not sequential equilibria.

Consider the behavior strategies of the form: $(p,1-p,1,0)$, $p \in \left(0,\frac{2}{3}\right)$.

Under such behavior strategies, the defendant knows that if he is offered a high settlement, that it is the "winning" plaintiff making the offer. Therefore, he will accept it. He will also reject any low settlement. Given this behavior strategy for the defendant, the "winning" plaintiff will have the profitable incentive to decrease the probability that he offers a high settlement. That is, $\forall p \in \left(0,\frac{2}{3}\right)$, $\exists p' \in \left(0,\frac{2}{3}\right)$ with $p' > p$ such that the plaintiff would strictly prefer to use the behavior strategy $(p',1-p',1,0)$ to $(p,1-p,1,0)$. That is, there is a profitable deviation. i.e. All behavior strategies of the form: $(p,1-p,1,0)$, $p \in \left(0,\frac{2}{3}\right)$ are not sequential equilibria.

Consider the behavior strategies of the form: $(p,1-p,0,1)$

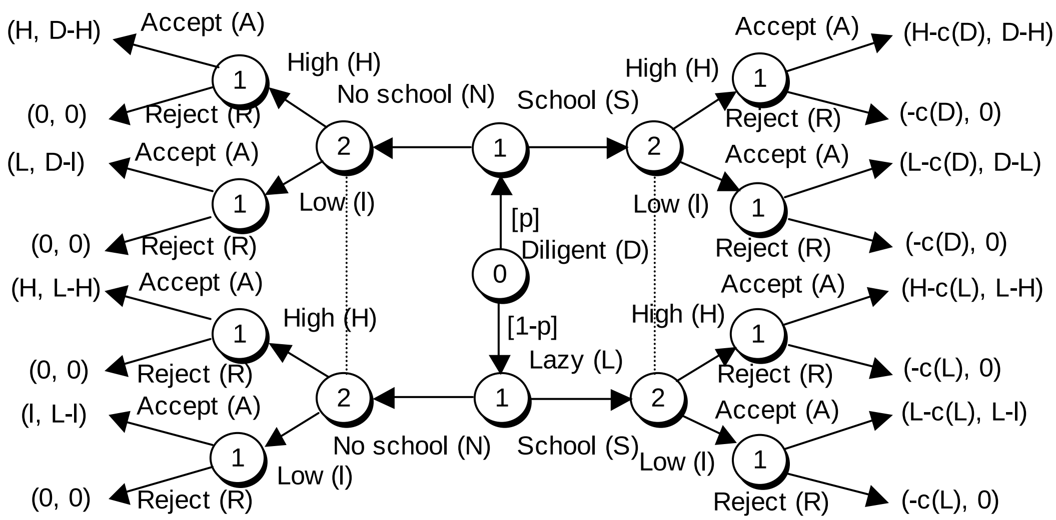
Under such behavior strategies, the defendant knows that if he is offered a low settlement, that it is the "winning" plaintiff making the offer. Therefore, he will accept it.

There are some values of $p$ for which he would accept the high settlement and some values of $p$ for which he would reject the low settlement. In either case, the "winning" plaintiff would have the profitable incentive to deviate and ask for the high settlement more of the time. That is, all behavior strategies of the form: $(p, 1-p, 0, 1)$ are not sequential equilibria.

**Education and Employment**

There are two players: a worker (player 1) and a firm (player 2). The worker has two types $\theta$: $D, L$, where $0 < L < D$ and $\Pr(D) = p$. The worker knows his type and chooses whether or not to get an education, at a cost $c(\theta)$. The firm observes the worker's education decision. The firm then offers a wage $w$: $H, l$. Finally, the worker accepts or rejects the wage. The profit to the firm from employing the worker is $\theta - w$, that is, education does not affect the worker's productivity. Suppose that $D - H > 0, L - H < 0$.

3.a Draw the game tree.




3.b For what values of $p$ do we get pooling, separating, and semi-separating Nash equilibria?

Pooling:
Assuming all wages are positive and using simple backward induction logic, we see that regardless of the wage offer by the firm, it is a strictly dominant strategy for the worker to accept.

Assuming that the low wage is lower than the high wage, it is a strictly dominant strategy for the firm to offer the low wage.

Assuming the cost of going to school is strictly positive, then we see that since going to school cannot possibly affect the wages received, it is a strictly dominant strategy not to go to school.

Thus, under these very reasonable assumptions, we have that $\forall p \in [0,1]$, there is a pooling equilibrium where both types of workers do not go to college. In addition,

$\forall p \in [0,1],$ we have that any strategy by either type of worker to become educated is not sequentially rational. Therefore, we cannot have any separating equilibria where either type of worker becomes educated. In addition, it is never rational for the firm to randomize between offering a high and a low wage at any of its nodes (as they could always place more weight on the low offer and increase their expected "utility"), so there are no semi-separating equilibria.

**Decreasing Absolute Risk Aversion**

A continuum of consumers has utility function $u(x) = 78x - x^2$. Each consumer has a $50\%$ chance of getting $x = 30$ and a $50\%$ chance of $x = 10$. Consider the following "mechanism:" a consumer that announces he has $x = 30$ pays $\tau$. A consumer that announces $x = 10$ receives a lottery with a $50\%$ chance of winning $g$ and a $50\%$ chance of winning $b$ where $.5g + .5b = \tau$. Suppose that "rich" consumers ($x = 30$) can lie and say that they are poor ($x = 10$). Find the mechanism that maximizes the expected utility of a consumer before he knows his type, subject to the constraint that the rich consumer does not wish to lie.

Under this mechanism, we have:

$$u(x = 30) = 78(30 - \tau) - (30 - \tau)^2$$

$$u(x = 10) = \frac{1}{2}\left(78(10 + g) - (10 + g)^2\right) + \frac{1}{2}\left(78(10 + b) - (10 + b)^2\right) \text{ where } .5g + .5b = \tau.$$

Simplifying, we have:

$$u(x = 30) = 1440 - 18\tau - \tau^2 \text{ and}$$

$$u(x = 10) = 680 - b^2 + 58\tau + 2b\tau - 2\tau^2$$

To maximize the expected utility of a consumer before he knows his type, we must maximize:

$$w = \frac{1}{2}u(x = 30) + \frac{1}{2}u(x = 10) \text{ by choosing } g, b, \text{and } \tau \text{ subject to the incentive}$$

compatibility constraint.

Simplifying, we have:

$$w = 1060 - \frac{b^2}{2} + 20\tau + b\tau - \frac{3\tau^2}{2}$$

(IC): $u(announce = 30 \mid \theta = 30) \geq u(announce = 10 \mid \theta = 30)$

$$\Rightarrow 78(30 - \tau) - (30 - \tau)^2 \geq \frac{1}{2}(78(30 + b) - (30 + b)^2) + \frac{1}{2}(78(30 + 2\tau - b) + (30 + 2\tau - b)^2)$$

$$\Rightarrow 1440 - 18\tau - \tau^2 \geq 1440 - b^2 + 18\tau + 2b\tau - 2\tau^2$$

$$\Rightarrow b^2 - 36\tau - 2b\tau + \tau^2 \geq 0$$

Therefore, our maximization problem is simply:

$$\max_{b,\tau} \frac{1}{2}u(x = 30) + \frac{1}{2}u(x = 10) \text{ subject to } b^2 - 36\tau - 2b\tau + \tau^2 \geq 0$$

$$L = 1060 - \frac{b^2}{2} + 20\tau + b\tau - \frac{3\tau^2}{2} + \lambda(b^2 - 36\tau - 2b\tau + \tau^2)$$

FOCs: (Assuming the incentive compatibility constraint is binding)

$(b)$: $\quad -b + \tau + \lambda(2b - 2\tau) \le 0 \qquad b \ge 0$

$(\tau)$: $\quad 20 + b - 3\tau + \lambda(2\tau - 2b - 36) \le 0 \qquad \tau \ge 0$

$(\lambda)$: $\quad b^2 - 36\tau - 2b\tau + \tau^2 = 0 \qquad \lambda > 0$

Assume $b > 0$ and $\tau > 0$: (These are the only non-degenerate cases)

$$\tau - b = \lambda(2\tau - 2b) \Rightarrow \lambda = \frac{\tau - b}{2(\tau - b)} = \frac{1}{2}$$

$$20 + b - 3\tau + \tau - b - 18 = 0 \Rightarrow 2 - 2\tau = 0 \Rightarrow \tau = 1$$

$$b^2 - 36\tau - 2b\tau + \tau^2 = b^2 - 36 - 2b + 1 = 0$$

$$\Rightarrow b^2 - 2b - 35 = 0 \Rightarrow (b-7)(b+5) = 0$$

$$\Rightarrow b = -5,\ 7.$$

Case I: Assume $b = -5$:

$$\frac{1}{2}b + \frac{1}{2}g = \tau \Rightarrow \frac{1}{2}(-5) + \frac{1}{2}g = 1 \Rightarrow -5 + g = 2 \Rightarrow g = 7$$

Thus, the mechanism with the parameters $(b, g, \tau) = (-5,7,1)$ works.

Case II: Assume $b = 7$:

$$\frac{1}{2}b + \frac{1}{2}g = \tau \Rightarrow \frac{1}{2}(7) + \frac{1}{2}g = 1 \Rightarrow 7 + g = 2 \Rightarrow g = -5$$

Thus, the mechanism with the parameters $(b, g, \tau) = (7,-5,1)$ works as well.

Much of the computation for this problem was completed in Mathematica. I have attached the input and output:

Is the expected utility from using this mechanism higher than the expected utility from not using any mechanism at all?

Expected utility from using mechanism:

$$\frac{1}{2}u(29) + \frac{1}{4}u(17) + \frac{1}{4}u(5) = \frac{1}{2}(78(29) - 29^2) + \frac{1}{4}(78(17) - 17^2) + \frac{1}{4}(78(5) - 5^2) = 1061$$

Expected utility without the mechanism:

$$\frac{1}{2}u(30) + \frac{1}{2}u(10) = \frac{1}{2}(78(30) - 30^2) + \frac{1}{2}(78(10) - 10^2) = 1060$$

Therefore, society is better off by adopting this mechanism.

Rich Consumer:

```
Expand[78*(30-t)-(30-t)^2]
```

$1440 - 18t - t^2$

**Poor Consumer**

$$\text{Expand}\left[\frac{1}{2}\,(78*(10+b)-(10+b)\verb|^|2)+\frac{1}{2}\,(78*(10+2*t-b)-(10+2*t-b)\verb|^|2)\right]$$

$680 - b^2 + 58t + 2bt - 2t^2$

**Expected utility of consumer**

$$\text{Expand}\left[\frac{1}{2}\left(1440-18t-t^2\right)+\frac{1}{2}\left(680-b^2+58t+2bt-2t^2\right)\right]$$

$$1060 - \frac{b^2}{2} + 20t + bt - \frac{3t^2}{2}$$

Rich Consumer who tells the truth

```
Expand[78*(30-t)-(30-t)^2]
```

$1440 - 18t - t^2$

**Rich Consumer who lies**

$$\text{Expand}\left[\frac{1}{2}\,(78*(30+b)-(30+b)\verb|^|2)+\frac{1}{2}\,(78*(30+2*t-b)-(30+2*t-b)\verb|^|2)\right]$$

$1440 - b^2 + 18t + 2bt - 2t^2$

Incentive Compatibility Constraint:

$$\text{Expand}\left[1440-18t-t^2-\left(1440-b^2+18t+2bt-2t^2\right)\right]$$

$b^2 - 36t - 2bt + t^2$

Expected Utility of Consumer with Incentive Compatibility Constraint:

$$1060-\frac{b^2}{2}+20t+bt-\frac{3t^2}{2}+l*\left(b^2-36t-2bt+t^2\right)$$

First order condition with respect to b:

$$\partial_b\left(1060-\frac{b^2}{2}+20t+bt-\frac{3t^2}{2}+l*\left(b^2-36t-2bt+t^2\right)\right)$$

-b+l (2 b-2 t)+t

First order condition with respect to t:

$$\partial_t\left(1060-\frac{b^2}{2}+20t+bt-\frac{3t^2}{2}+l*\left(b^2-36t-2bt+t^2\right)\right)$$

20+b-3 t+l (-36-2 b+2 t)

First order condition with respect to l:

$$\partial_l\left(1060-\frac{b^2}{2}+20t+bt-\frac{3t^2}{2}+l*\left(b^2-36t-2bt+t^2\right)\right)$$

$b^2 - 36t - 2bt + t^2$

Assuming t>0 and b>0, we have:

```
Solve[b^2-2*b-35==0,b]
```

{{b→-5},{b→7}}

**Moral Hazard**

There are 2 states of the world $s \in \{1,2\}$ and 2 possible actions $a \in \{1,2\}$. A risk neutral principal observes only the state and not the action of the agent he hires. The net gain of an agent if he is paid *w* and takes action *a* is $v(w) - c(a)$, where $c(1) < c(2)$. Under action *a* the probability of state *s* is $p_s(a)$, where $p_2(a)$ is increasing in *a*. The agent's reservation utility is 0. The output (received by the principal) is $y_s$ where $y_2 > y_1$.

5.a The principal wishes to induce action $a = 2$ and only "downward" constraints of pretending lower cost are potentially binding. What condition is sufficient for the optimal incentive scheme $w_1, w_2$ to be monotonic? Prove your claim.

I have changed notation slightly. Let $c(1) \equiv c_1$ and $c(2) \equiv c_2$. We know that the principal wishes to induce action $m = 2$. It would be optimal for the principal to do so at the lowest cost. That is, we have the following optimization problem:

$\min_{w_1, w_2} w_1 p_1(2) + w_2 p_2(2)$ subject to

$(IC)$ $p_1(2)[v(w_1) - c_2] + p_2(2)[v(w_2) - c_2] \geq p_1(1)[v(w_1) - c_1] + p_2(1)[v(w_2) - c_1]$

$(IR)$ $p_1(2)[v(w_1) - c_2] + p_2(2)[v(w_2) - c_2] \geq U^*$ where $U^*$ is the agent's reservation utility.

Setting up the Lagrangian for this problem, we have:

$$L = -w_1 p_1(2) - w_2 p_2(2) + \lambda\{p_1(2)[v(w_1) - c_2] + p_2(2)[v(w_2) - c_2] - p_1(1)[v(w_1) - c_1] - p_2(1)[v(w_2) - c_1]\} + \mu\{p_1(2)[v(w_1) - c_2] + p_2(2)[v(w_2) - c_2] \geq U^*\}$$

The first order conditions are:

$$(w_1): - p_1(2) + \lambda p_1(2)v'(w_1) - \lambda p_1(1)v'(w_1) + \mu p_1(2)v'(w_1) \leq 0 \qquad w_1 \geq 0$$

$$(w_2): - p_2(2) + \lambda p_2(2)v'(w_2) - \lambda p_2(1)v'(w_2) + \mu p_2(2)v'(w_2) \leq 0 \qquad w_2 \geq 0$$

$$(\lambda): p_1(2)[v(w_1) - c_2] + p_2(2)[v(w_2) - c_2] \geq p_1(1)[v(w_1) - c_1] + p_2(1)[v(w_2) - c_1] \qquad \lambda \geq 0$$

$$(\mu): p_1(2)[v(w_1) - c_2] + p_2(2)[v(w_2) - c_2] \geq U^* \qquad \mu \geq 0$$

Assuming $w_1 > 0$ and $w_2 > 0$,

From $(w_1)$ we have:

$$v'(w_1)[\lambda p_1(2) - \lambda p_1(1) + \mu p_1(2)] = p_1(2)$$

$$\Rightarrow v'(w_1) = \frac{p_1(2)}{\lambda[p_1(2) - p_1(1)] + \mu p_1(2)}$$

$$\Rightarrow \frac{1}{v'(w_1)} = \frac{\mu p_1(2)}{p_1(2)} + \frac{\lambda[p_1(2) - p_1(1)]}{p_1(2)} = \mu + \lambda\left[1 - \frac{p_1(1)}{p_1(2)}\right]$$

From $(w_2)$ we have:

$$v'(w_2)[\lambda p_2(2) - \lambda p_2(1) + \mu p_2(2)] = p_2(2)$$

$$\Rightarrow v'(w_2) = \frac{p_2(2)}{\lambda[p_2(2) - p_2(1)] + \mu p_2(2)}$$

$$\Rightarrow \frac{1}{v'(w_2)} = \frac{\mu p_2(2)}{p_2(2)} + \frac{\lambda[p_2(2) - p_2(1)]}{p_2(2)} = \mu + \lambda\left[1 - \frac{p_2(1)}{p_2(2)}\right]$$

From this, we have:

$w_2{*} \geq w_1{*} \Leftrightarrow v'(w_2{*}) \leq v'(w_1{*})$ if we assume that the agent is risk averse (by concavity of utility functions of risk averse individuals

$$\Leftrightarrow \frac{1}{v'(w_2{*})} \geq \frac{1}{v'(w_1{*})} \Leftrightarrow \frac{1}{v'(w_2{*})} - \frac{1}{v'(w_1{*})} \geq 0$$

$$\Leftrightarrow \mu + \lambda\left[1 - \frac{p_1(1)}{p_1(2)}\right] - \mu - \lambda\left[1 - \frac{p_2(1)}{p_2(2)}\right] \geq 0$$

$$\Leftrightarrow \lambda\left[1 - \frac{p_1(1)}{p_1(2)} - 1 + \frac{p_2(1)}{p_2(2)}\right] = \lambda\left[\frac{p_2(1)}{p_2(2)} - \frac{p_1(1)}{p_1(2)}\right] \geq 0$$

$$\Leftrightarrow \frac{p_2(1)}{p_2(2)} \geq \frac{p_1(1)}{p_1(2)}$$

Therefore, the sufficient conditions for $w_2{*} \geq w_1{*}$ are that $\frac{p_2(1)}{p_2(2)} \geq \frac{p_1(1)}{p_1(2)}$ holds and that the utility functions are concave.

5.b Suppose $v(w) = 1 - e^{-\gamma w}$. What more can be said about $w_2 - w_1$?

We have that $MRS_{2,1} = \frac{v'(w_2)}{v'(w_1)} = \frac{\gamma e^{-\gamma w_2}}{\gamma e^{-\gamma w_1}} = e^{-\gamma(w_2 - w_1)}$

From part $a$, we also have: $MRS_{2,1} = \dfrac{\frac{1}{v'(w_1)}}{\frac{1}{v'(w_2)}} = \dfrac{\mu + \lambda\left[1 - \frac{p_1(1)}{p_1(2)}\right]}{\mu + \lambda\left[1 - \frac{p_2(1)}{p_2(2)}\right]}$

Therefore, we have: $e^{-\gamma(w_2 - w_1)} = \dfrac{\mu + \lambda\left[1 - \frac{p_1(1)}{p_1(2)}\right]}{\mu + \lambda\left[1 - \frac{p_2(1)}{p_2(2)}\right]}$

$$\Rightarrow -\gamma(w_2 - w_1) = \ln\left[\mu + \lambda\left(1 - \frac{p_1(1)}{p_1(2)}\right)\right] - \ln\left[\mu + \lambda\left(1 - \frac{p_2(1)}{p_2(2)}\right)\right]$$

$$\Rightarrow w_2 - w_1 = \frac{1}{\gamma}\left\{\ln\left[\mu + \lambda\left(1 - \frac{p_2(1)}{p_2(2)}\right)\right] - \ln\left[\mu + \lambda\left(1 - \frac{p_1(1)}{p_1(2)}\right)\right]\right\}$$

To gain some intuition about this result, this rather complicated equation can be written as: $w_2 - w_1 = \frac{1}{\gamma}c$

We know that $A(w) = \frac{-v''(w)}{v'(w)} = \frac{\gamma^2 e^{-\gamma w}}{\gamma e^{-\gamma w}} = \gamma.$

That is, the wage differential $w_2 - w_1$ is a decreasing function absolute risk aversion.

5.c For this special case, discuss the effect on the optimal incentive scheme if there is a change in the agent's reservation utility, assuming the principal still wants to induce the action $a = 2$.

The following lemma will be helpful in this problem:

Lemma 5.c.1:

If $\frac{a}{b} \le 1$ and $\alpha > 0,$ then $\frac{\alpha + a}{\alpha + b} \ge \frac{a}{b}.$

Proof of lemma 5.c.1:

Let $\frac{a}{b} \le 1 \Rightarrow a \le b \Rightarrow \alpha a \le \alpha b$ since $\alpha > 0$

$\Rightarrow \alpha a + ab \le \alpha b + ab \Rightarrow a(\alpha + b) \le b(\alpha + a) \Rightarrow \frac{a}{b} \le \frac{\alpha + a}{\alpha + b}$. Q.E.D.

During a study group session, Hong Feng proved that in this problem, the individual rationality constraint must be binding. I will not duplicate his proof here (since I can't remember the details), but I will use it in the following conjecture: As the reservation utility increases, $\mu$, the shadow price of reservation utility must increase.

I will assume the conditions from part a hold. Since from part b, we know that:

$w_2 - w_1 = \frac{1}{\gamma}\ln\left[\frac{\mu + \lambda\left(1 - \frac{p_2(1)}{p_2(2)}\right)}{\mu + \lambda\left(1 - \frac{p_1(1)}{p_1(2)}\right)}\right]$ the following applies:

$$\frac{p_2(1)}{p_2(2)} \geq \frac{p_1(1)}{p_1(2)} \Rightarrow 1-\frac{p_2(1)}{p_2(2)} \leq 1-\frac{p_1(1)}{p_1(2)} \Rightarrow \lambda\left[1-\frac{p_2(1)}{p_2(2)}\right] \leq \lambda\left[1-\frac{p_1(1)}{p_1(2)}\right]$$

$$\Rightarrow \mu+\lambda\left[1-\frac{p_2(1)}{p_2(2)}\right] \leq \mu+\lambda\left[1-\frac{p_1(1)}{p_1(2)}\right] \Rightarrow \frac{\mu+\lambda\left(1-\frac{p_2(1)}{p_2(2)}\right)}{\mu+\lambda\left(1-\frac{p_1(1)}{p_1(2)}\right)} \leq 1$$

Therefore, by lemma 5.c.1 and the conjecture from earlier, we have that an increase in reservation utility would lead to an increase in $\mu$ from $\mu$ to $\mu+\varepsilon(\Delta U^*)$, which would result in: $\frac{\mu+\varepsilon(\Delta U^*)+\lambda\left(1-\frac{p_2(1)}{p_2(2)}\right)}{\mu+\varepsilon(\Delta U^*)+\lambda\left(1-\frac{p_1(1)}{p_1(2)}\right)} \geq \frac{\mu+\lambda\left(1-\frac{p_2(1)}{p_2(2)}\right)}{\mu+\lambda\left(1-\frac{p_1(1)}{p_1(2)}\right)}$. Therefore, an increase in reservation utility would lead to an increase in: $\frac{1}{\gamma}\ln\left[\frac{\mu+\lambda\left(1-\frac{p_2(1)}{p_2(2)}\right)}{\mu+\lambda\left(1-\frac{p_1(1)}{p_1(2)}\right)}\right]$, which from part b would lead to an increase in the wage differential $w_2^*-w_1^*$.

The economic interpretation of this could perhaps be that an individual with a higher reservation utility would be willing to "hold out for more" before he put forth the extra effort.

5.d Is there a change in the agent's reservation utility that would lead the principal to prefer to induce the action $a=1$?

The utility of the principal when he induces action $a=1$ is:
$\pi(a=1)=p_1(1)[y_1-w_1]+[1-p_1(1)][y_2-w_2]$

The utility of the principal when he induces action $a=2$ is:
$\pi(a=2)=p_1(2)[y_1-w_1]+[1-p_1(2)][y_2-w_2]$

When is it optimal for the principal to induce action $a=2$? When $\pi(a=2)\geq\pi(a=1)$.
$\Leftrightarrow p_1(2)[y_1-w_1]+[1-p_1(2)][y_2-w_2] \geq p_1(1)[y_1-w_1]+[1-p_1(1)][y_2-w_2]$
$\Leftrightarrow [y_1-w_1][p_1(2)-p_1(1)]+[y_2-w_2][1-p_1(2)-1+p_1(1)] \geq 0$
$\Leftrightarrow [y_1-w_1][p_1(2)-p_1(1)] \geq [y_2-w_2][p_1(2)-p_1(1)]$
$\Leftrightarrow \frac{y_1-w_1}{y_2-w_2} \leq \frac{p_1(2)-p_1(1)}{p_1(2)-p_1(1)}=1$ where I reversed the inequality sign because the assumption that $p_2(2)>p_2(1) \Rightarrow p_1(2)<p_1(1) \Rightarrow p_1(2)-p_2(1)<0$

$\Leftrightarrow\; y_1 - w_1 \le y_2 - w_2 \Leftrightarrow w_2 - w_1 \le y_2 - y_1$

That is, it is optimal to induce action $a = 2$ if the wage differential is less than or equal to the difference in outputs in the two states. (Assuming the prices of the outputs have been normalized to 1).

From part c, we know that an increase in the reservation utility leads to an increase in the wage differential. If the reservation utility increases enough, the wage differential could potentially increase enough so as to cause: $w_2 - w_1 > y_2 - y_1$, in which case it would be optimal for the principal to induce action $a = 1$.

5.e Suppose that the agent's utility is $u(w,a)$ and is not separable. Is it possible to induce the agent to use $a = 2$ for arbitrarily large reservation utilities?

If the utility function $u(w,a)$ is such that the optimal wage differential is not an increasing function of reservation utilities, then this could be the case.

Consider a general non separable utility function $u(w,a)$ and the principal's optimal mechanism problem:

$$\min_{w_1,w_2} w_1 p_1(2) + w_2 p_2(2) \text{ subject to}$$
$$(IC) \quad p_1(2)u(w_1,2) + p_2(2)u(w_2,2) \ge p_1(1)u(w_1,1) + p_2(1)u(w_2,1)$$
$$(IR) \quad p_1(2)u(w_1,2) + p_2(2)u(w_2,2) \ge U^* \text{ where } U^* \text{ is the reservation utility.}$$

Setting up the Lagrangian for this problem, we have:

$$L = -w_1 p_1(2) - w_2 p_2(2) + \lambda[p_1(2)u(w_1,2) + p_2(2)u(w_2,2) - p_1(1)u(w_1,1) - p_2(1)u(w_2,1)]$$
$$+ \mu[p_1(2)u(w_1,2) + p_2(2)u(w_2,2) - U^*]$$

The first order conditions are therefore:

$$(w_1): - p_1(2) + \lambda p_1(2)u_1(w_1,2) - \lambda p_1(1)u_1(w_1,1) + \mu p_1(2)u_1(w_1,2) \ge 0 \qquad w_1 \ge 0$$
$$(w_2): - p_2(2) + \lambda p_2(2)u_1(w_2,2) - \lambda p_2(1)u_1(w_2,1) + \mu p_2(2)u_1(w_2,2) \ge 0 \qquad w_2 \ge 0$$
$$(\lambda): p_1(2)u(w_1,2) + p_2(2)u(w_2,2) \ge p_1(1)u(w_1,1) + p_2(1)u(w_2,1) \qquad \lambda \ge 0$$
$$(\mu): p_1(2)u(w_1,2) + p_2(2)u(w_2,2) \ge U^* \qquad \mu \ge 0$$

Proceeding as in part a, we have:

From $(w_1)$: $$u_1(w_1^*,2) = \frac{p_1(2)}{\lambda(p_1(2) - p_1(1)) + \mu p_1(2)} \Rightarrow \frac{1}{u_1(w_1^*,2)} = \mu + \lambda\left[1 - \frac{p_1(1)}{p_1(2)}\right]$$

From $(w_2)$: $$u_1(w_2^*,2) = \frac{p_2(2)}{\lambda(p_2(2) - p_2(1)) = \mu p_2(2)} \Rightarrow \frac{1}{u_1(w_2^*,2)} = \mu + \lambda\left[1 - \frac{p_2(1)}{p_2(2)}\right]$$

Consider the non separable utility function $u(w,a)=\ln(w-a)$

For this function, we have:

$$u_1(w_1*,2)=\frac{1}{w_1*-2} \text{ and } u_2(w_2,2)=\frac{1}{w_2-2}$$

$$\Rightarrow \frac{1}{u_1(w_1*,2)}=w_1*-2 \text{ and } \frac{1}{u_1(w_2,2)}=w_2-2$$

Therefore, we have:

$$\frac{1}{u_1(w_2*,2)}-\frac{1}{u_2(w_1*,2)}=w_2*-w_1* \text{ and}$$

$$\frac{1}{u_1(w_2*,2)}-\frac{1}{u_2(w_1*,2)}=\mu+\lambda\left[1-\frac{p_2(1)}{p_2(2)}\right]-\mu-\lambda\left[1-\frac{p_1(1)}{p_1(2)}\right]=\lambda\left[\frac{p_2(1)}{p_2(2)}-\frac{p_1(1)}{p_1(2)}\right]$$

Combining these, we have:

$w_2*-w_1*=\lambda\left[\frac{p_2(1)}{p_2(2)}-\frac{p_1(1)}{p_1(2)}\right]$. Therefore, in this case, the optimal wage differential is not a function of $\mu$ and is therefore not a function of the reservation utility. In this case, it could then be possible for the principal to induce $a=2$ for arbitrarily large reservation utilities. (This would also hold for many separable utility functions – just not the specific example we were given in part b)

**Adverse Selection**

Consider a continuum of ex ante identical individuals with utility function for consumption *c* of $-e^{-c}$. Ex post, two states are possible. In state 1 the endowment is 2. In state 2 the endowment is 0. What is the first best allocation? Suppose that the state is privately known. Show that there is no incentive compatible ex ante exclusive contract that gives the low endowment type more utility than at autarky.

Part I:
I am not certain what exactly a "first best allocation" is, but I will assume that it is the allocation of the scarce resources that maximizes the sum of the individuals' utility.

In addition, I will assume that $\Pr\{state = 1\} = p$ and $\Pr\{state = 2\} = 1 - p$.

Given this probability distribution over the states, which I will assume to be iid among the consumers, the total resources available to the society is 2*p*. (As *p* of the consumers will receive an endowment of 2 and *1-p* of the consumers will receive an endowment of 0.)

Since the consumers are risk averse, we have that they would strictly prefer the mean consumption to any sort of randomizing device. (See Jensen's inequality) If we normalize the continuum of individuals to an interval of length one, it follows that the first best allocation would give each consumer 2*p* units of consumption.

Part II:
In an autarky, the low type receives an endowment of 0. Any ex ante exclusive contract (I am speculating as to the definition) requires that those who announce that they are state 1 consumers reduce their consumption in order to give to the state 2 consumers. However, this occurs after the types have already been drawn. Therefore, no one who is a state 1 individual would have any incentive to announce that he/she is a state 1 individual. This is clearly not incentive compatible under any revelation-style mechanism

In other words, we have:
$Eu(announce = 2 \mid state = 1) = -e^{-2-\varepsilon}$, where $\varepsilon$ is the expected transfer from the mechanism.
$Eu(announce = 1 \mid state = 1) = -e^{-2+\varepsilon}$.
Clearly, $Eu(announce = 1 \mid state = 1) < Eu(announce = 2 \mid state = 1)$, which violates the incentive compatibility constraint.