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Problem Set 1

Static Games

For each of the following games, find 1) all weak and strict dominant strategy equilibria, 2) apply iterated strict dominance, 3) find all pure and mixed Nash equilibria, 4) indicate which Nash equilibria are trembling hand perfect and why

1.a

	L(q)	R(1-q)
U(p)	2 1	0 0
D(1-p)	0 0	1 2

1) There are no weak or strict dominant strategy equilibria.
2) No pure strategies are strictly dominated
3)

	L(q)	R(1-q)
U(p)	2 1	0 0
D(1-p)	0 0	1 2

Since *U* is a best response for player I to *L* and *L* is a best response for player II to *U*, the strategy profile $\sigma = (U, L)$ is a pure Nash Equilibrium. Since *D* is a best response for player I to *R* and *R* is a best response for player II to *D*, the strategy profile $\sigma = (D, R)$ is a pure Nash Equilibrium.

Let $\sigma_2 = (q, 1-q)$.
$u_1(U, \sigma_2) = 2(q) + 0(1-q) = 2q$
$u_1(D, \sigma_2) = 0(q) + 1(1-q) = 1-q$. In equilibrium, $u_1(U, \sigma_2) = u_1(D, \sigma_2)$
$\Rightarrow 2q = 1-q \Rightarrow 3q = 1 \Rightarrow q = \frac{1}{3}$.
Let $\sigma_1 = (p, 1-p)$.
$u_2(L, \sigma_1) = 1(p) + 0(1-p) = p$
$u_2(R, \sigma_1) = 0(p) + 2(1-p) = 2-2p$. In equilibrium, $u_2(L, \sigma_1) = u_2(R, \sigma_1)$
$\Rightarrow p = 2-2p \Rightarrow 3p = 2 \Rightarrow p = \frac{2}{3}$.
Thus, $\sigma = (\sigma_1, \sigma_2) = \left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$ is a mixed Nash Equilibrium.

4) (*U*,*L*) is trembling hand perfect, because it is a strict NE.
(*D*,*R*) is trembling hand perfect, because it is a strict NE.

$\left(\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{2}{3}\right)\right)$ is trembling hand perfect, because it is completely mixed.

1.b

	L		R	
U	6	6	0	7
D	7	0	1	1

1) (*D*,*R*) is a strict dominant strategy equilibrium.

2) Since $(u_1(D,L),u_1(D,R))=(7,1)>>(6,0)=(u_1(U,L),u_1(U,R))$, the strategy *U* is strictly dominated:

	L		R	
U	[redacted]	[redacted]	[redacted]	[redacted]
D	7	0	1	1

Since $u_2(R,D)=1>0=u_2(L,D)$, the strategy *R* is strictly dominated:

	L		R	
U	[redacted]	[redacted]	[redacted]	[redacted]
D	[redacted]	[redacted]	1	1

The remaining strategy profile (*D,R*) is thus the Nash Equilibrium of the reduced form game.

3) A strategy profile is a Nash Equilibrium of a game if and only if it is a Nash Equilibrium of the game where the dominated strategies have been removed by iterated strict dominance. Therefore, (D,R) is a pure Nash Equilibrium. Since it is the only remaining strategy profile, there can be no non-degenerate mixed strategy profiles.

4) (D,R) is a strict N.E. Therefore, it is trembling hand perfect.

1.c

	L		M		R	
U	3	3	2	2	1	1
M	2	2	1	1	0	8
D	1	1	8	0	0	0

1) There are no weak or strict dominant strategy equilibria

2) Since $(u_1(M,L),u_1(M,M),u_1(M,R)) << (u_1(U,L),u_1(U,M),u_1(U,R)),$ it follows that *M* is a strictly dominated strategy for player I:

	L		M		R	
U	3	3	2	2	1	1
M	■	■	■	■	■	■
D	1	1	8	0	0	0

Since $(u_2(U,M),u_2(D,M)) << (u_2(U,L),u_2(D,L)),$ it follows that *M* is a strictly dominated strategy for player II:

	L		M		R	
U	3	3	■	■	1	1
M	■	■	■	■	■	■
D	1	1	■	■	0	0

Since $(u_1(D,L),u_1(D,R)) << (u_1(U,L),u_1(U,R)),$ it follows that *D* is a strictly dominated strategy for player I:

	L		M		R	
U	3	3	■	■	1	1
M	■	■	■	■	■	■
D	■	■	■	■	■	■

Since $u_2(U,R) < u_2(U,L),$ it follows that *R* is a strictly dominated strategy for player II:

	L		M		R	
U	3	3	■	■	■	■
M	■	■	■	■	■	■
D	■	■	■	■	■	■

The remaining strategy profile (U,L) is thus the Nash equilibrium of the reduced form game.

3) A strategy profile is a Nash Equilibrium of a game if and only if it is a Nash Equilibrium of the game where the dominated strategies have been removed by iterated strict dominance. Therefore, (U,L) is a pure Nash Equilibrium. Since it is the only remaining strategy profile, there can be no non-degenerate mixed strategy profiles.

4) Since (U,L) is a strict Nash Equilibrium, it is trembling hand perfect.

1.d

	L		R	
U	1	3	1	3
D	0	0	2	0

1) There are no weak or strict dominant strategy equilibria.

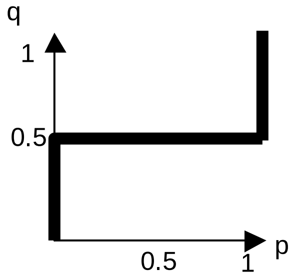
2) There are no strictly dominated strategies in this game.

3)

	L		R	
U	1	3	1	3
D	0	0	2	0

(U,L) and (D,R) are pure strategy Nash Equilibria.

For this game, it is easiest to see all the mixed strategies by constructing a best response correspondence: Assume player I plays *U* with probability *p* and player II plays *L* with probability *q*.



Player I's best response correspondence is drawn. Since player II is completely indifferent between *L* and *R* for all of player I's strategies, the entire graph is player II's best response correspondence.

Therefore, the mixed strategy profiles that are a mutual best response (i.e. Nash Equilibrium) are:

$$\left((p,1-p),\left(\frac{1}{2},\frac{1}{2}\right)\right),\ p\in[0,1]$$

$$((0,1),(q,1-q)),\ q\in\left(0,\frac{1}{2}\right)$$

$$((1,0),(q,1-q)),\ q\in\left(\frac{1}{2},1\right)$$

4) Trembling hand perfection:

Since player II is always indifferent between her two strategies, she will never have any incentive to deviate from her strategy when we introduce trembles. Therefore, only player I's actions are relevant:

Consider the NE (U,L):

Construct the "tremble" $\sigma^n=\left(\left(1-\frac{1}{2n},\frac{1}{2n}\right),\left(1-\frac{1}{2n},\frac{1}{2n}\right)\right),\ n=2,3,...$

$$u_1\left(U,\left(1-\frac{1}{2n},\frac{1}{2n}\right)\right)=(1)\left(1-\frac{1}{2n}\right)+(1)\left(\frac{1}{2n}\right)=1$$

$$u_1\left(D,\left(1-\frac{1}{2n},\frac{1}{2n}\right)\right)=(0)\left(1-\frac{1}{2n}\right)+(2)\left(\frac{1}{2n}\right)=\frac{1}{n}<1\ \forall n=2,3,...$$

Thus, player I has no incentive to deviate.

$$u_2\left(L,\left(1-\frac{1}{2n},\frac{1}{2n}\right)\right)=u_2\left(R,\left(1-\frac{1}{2n},\frac{1}{2n}\right)\right)$$

Thus, player II has no incentive to deviate. Therefore, (U,L) is trembling hand perfect.

Consider the NE (D,R):

Construct the "tremble" $\sigma^n=\left(\left(\frac{1}{2n},1-\frac{1}{2n}\right),\left(\frac{1}{2n},1-\frac{1}{2n}\right)\right),\ n=2,3,\ldots$

$$u_1\left(U,\left(\frac{1}{2n},1-\frac{1}{2n}\right)\right)=(1)\left(\frac{1}{2n}\right)+(1)\left(1-\frac{1}{2n}\right)=1$$

$$u_1\left(D,\left(\frac{1}{2n},1-\frac{1}{2n}\right)\right)=(0)\left(\frac{1}{2n}\right)+(2)\left(1-\frac{1}{2n}\right)=2-\frac{1}{n}>1\ \forall n=2,3,\ldots$$

Thus, player I has no incentive to deviate.

$$u_2\left(L,\left(\frac{1}{2n},1-\frac{1}{2n}\right)\right)=u_2\left(R,\left(\frac{1}{2n},1-\frac{1}{2n}\right)\right)$$

Thus, player II has no incentive to deviate. Therefore, (D,R) is trembling hand perfect.

Consider the NE of the form $\left((p,1-p),\left(\frac{1}{2},\frac{1}{2}\right)\right),\ p\in[0,1]$:

For all of these NE, player I is indifferent between playing *U* and playing *D*. $\forall p\in[0,1]$, player II is also indifferent between playing *L* and *R*. Since player II's strategies are already mixed, we would only have to construct a sequence of "trembles" for player II, but since player II does not care what player I plays, these "trembles" will clearly not give her any incentive to deviate. Therefore, these NE are trembling hand perfect.

Consider the NE of the form $((0,1),(q,1-q)),\ q\in\left(0,\frac{1}{2}\right)$:

Construct the sequence of "trembles" $\sigma^n=\left(\left(\frac{1}{2n},1-\frac{1}{2n}\right),(q,1-q)\right)$:

For all $q\in\left(0,\frac{1}{2}\right)$, player I strictly prefers to play D. Since player II's strategy is already mixed, the trembles are trivial. Therefore, player I will not deviate.

$$u_2\left(L,\left(\frac{1}{2n},1-\frac{1}{2n}\right)\right)=u_2\left(R,\left(\frac{1}{2n},1-\frac{1}{2n}\right)\right)$$

Player II has no incentive to deviate. Therefore, all these NE are trembling hand perfect.

Consider the NE of the form $((1,0),(q,1-q)),\ q\in\left(\frac{1}{2},1\right)$:

Construct the sequence of "trembles" $\sigma^n=\left(\left(1-\frac{1}{2n},\frac{1}{2n}\right),(q,1-q)\right)$

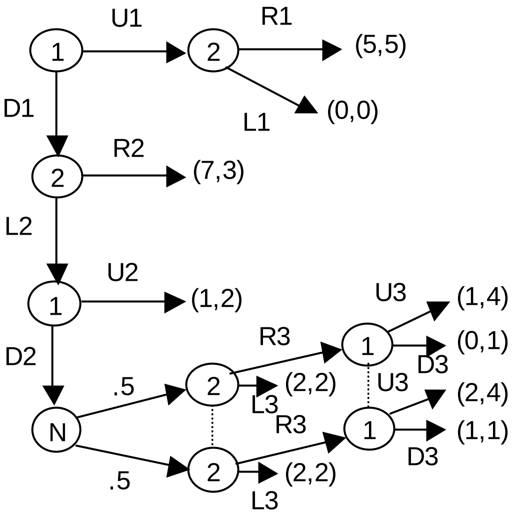
For all $q\in\left(\frac{1}{2},1\right)$, player I strictly prefers to play D. Since player II's strategy is already mixed, the trembles are trivial. Therefore, player I will not deviate.

$$u_2\left(L,\left(1-\frac{1}{2n},\frac{1}{2n}\right)\right)=u_2\left(R,\left(1-\frac{1}{2n},\frac{1}{2n}\right)\right)$$

Player II has no incentive to deviate. Therefore, all these NE are trembling hand perfect.

Dynamic Games

In the game below find 1) the normal form 2) all pure and mixed Nash equilibria 3) all subgame perfect equilibria



1)

	L1L2L3		L1L2R3		L1R2L3		L1R2R3		R1L2L3		R1L2R3		R1R2L3		R1R2R3	
U1U2U3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
U1U2D3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
U1D2U3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
U1D2D3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
D1U2U3	1	2	1	2	7	3	7	3	1	2	1	2	7	3	7	3
D1U2D3	1	2	1	2	7	3	7	3	1	2	1	2	7	3	7	3
D1D2U3	2	2	1.5	4	7	3	7	3	2	2	1.5	4	7	3	7	3
D1D2D3	2	2	0.5	1	7	3	7	3	2	2	0.5	1	7	3	7	3

2)

	L1L2L3		L1L2R3		L1R2L3		L1R2R3		R1L2L3		R1L2R3		R1R2L3		R1R2R3	
U1U2U3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
U1U2D3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
U1D2U3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
U1D2D3	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5	5
D1U2U3	1	2	1	2	7	3	7	3	1	2	1	2	7	3	7	3
D1U2D3	1	2	1	2	7	3	7	3	1	2	1	2	7	3	7	3
D1D2U3	2	2	1.5	4	7	3	7	3	2	2	1.5	4	7	3	7	3
D1D2D3	2	2	0.5	1	7	3	7	3	2	2	0.5	1	7	3	7	3

Pure NE: (U1U2U3,R1L2L3), (U1U2U3,R1L2R3), (U1U2D3,R1L2L3), (U1U2D3,R1L2R3), (U1D2U3,R1L2L3), (U1D2U3,R1L2R3), (U1D2D3,R1L2L3), (U1D2D3,R1L2R3), (D1U2U3,L1R2L3), (D1U2U3,L1R2R3), (D1U2U3,R1R2L3), (D1U2U3,R1R2R3), (D1U2D3,L1R2L3), (D1U2D3,L1R2R3), (D1U2D3,R1R2L3), (D1U2D3,R1R2R3), (D1D2U3,L1L2R3), (D1D2D3,L1R2L3), (D1D2D3,L1R2R3), (D1D2D3,R1R2L3), (D1D2D3,R1R2R3)

Mixed NE: In order to find the mixed Nash equilibria, it is necessary to first construct the reduced normal form by eliminating any equivalent strategies:

	L1L2L3		L1L2R3		L1R2		R1L2L3		R1L2R3		R1R2	
U1	0	0	0	0	0	0	5	5	5	5	5	5
D1U2	1	2	1	2	7	3	1	2	1	2	7	3
D1D2U3	2	2	1.5	4	7	3	2	2	1.5	4	7	3
D1D2D3	2	2	0.5	1	7	3	2	2	0.5	1	7	3

Clearly, L1L2L3 is a dominated strategy for player II. (It is dominated by R1R2):

	L1L2R3		L1R2		R1L2L3		R1L2R3		R1R2	
U1	0	0	0	0	5	5	5	5	5	5
D1U2	1	2	7	3	1	2	1	2	7	3
D1D2U3	1.5	4	7	3	2	2	1.5	4	7	3
D1D2D3	0.5	1	7	3	2	2	0.5	1	7	3

My procedure for finding the mixed Nash equilibria is as follows:

1) Find a strategy that is weakly dominated.

2) Partition the set of possible supports into two sets: one in which each support includes the pure strategy that, if it were given a positive probability, would cause the other player not to play the weakly dominated strategy; the other in which each support does not include the aforementioned pure strategy

3) Continue this process to eliminate the “unplausible” supports.

4) Given the set of “plausible” supports, determine the mixed Nash equilibria using the method detailed in the first discussion section.

1) I notice that L1L2R3 is weakly dominated by R1L2R3 and that L1R2 is weakly dominated by R1R2.

2) Suppose $\text{U1} \in \text{supp}(\sigma_1)$. Then the expected utility for player II of playing L1L2R3 is strictly less than the expected utility of playing R1L2R3. Therefore, L1L2R3 and R1L2R3 will not be in the support of σ_2. This suggests that we can partition the set of possible supports into sets A and B, where $A = \{\text{supports} \,|\, \text{U1} \in \text{supp}(\sigma_1)$ and L1L2R3,R1L2R3 $\notin \text{supp}(\sigma_2)\,\}$ and $B = \{\text{supports} \,|\, \text{U1} \notin \text{supp}(\sigma_1)\}$.

A) Consider the set of supports $A = \{\text{supports} \,|\, \text{U1} \in \text{supp}(\sigma_1)$ and L1L2R3,R1L2R3 $\notin \text{supp}(\sigma_2)\,\}$. The "Feng-Powell reduced form" (for lack of a better name) of the game which does not include the strategies which are not in the support of one of the players is:

	R1L2L3		R1L2R3		R1R2	
U1	5	5	5	5	5	5
D1U2	1	2	1	2	7	3
D1D2U3	2	2	1.5	4	7	3
D1D2D3	2	2	0.5	1	7	3

A1) I notice that R1L2L3 is weakly dominated by R1R2.

A2) Suppose at least one of D1U2, D1D2U3, D1D2D3 $\in \text{supp}(\sigma_1)$. Then R1L2L3 $\notin \text{supp}(\sigma_2)$. This suggests that we can partition the set *A* into the sets *C* and *D*, where $A \supset C = \{$supports|at least one of D1U2, D1D2U3, D1D2D3 $\in \text{supp}(\sigma_1)$, R1L2L3 $\notin \text{supp}(\sigma_2)\}$ and $A \supset D = \{$supports|D1U2 $\notin \text{supp}(\sigma_1)$, D1D2U3 $\notin \text{supp}(\sigma_1)$, D1D2D3 $\notin \text{supp}(\sigma_1)\,\}$.

B) Consider the set of supports $B = \{$supports|U1 $\notin \text{supp}(\sigma_1)\,\}$. The "Feng-Powell reduced form" of the game which does not include the strategies which are not in the support of one of the players is:

	L1L2R3		L1R2		R1L2L3		R1L2R3		R1R2	
D1U2	1	2	7	3	1	2	1	2	7	3
D1D2U3	1.5	4	7	3	2	2	1.5	4	7	3
D1D2D3	0.5	1	7	3	2	2	0.5	1	7	3

B1) I notice that D1U2 is weakly dominated by D1D2U3 and D1D2D3 is weakly dominated by D1D2U3.

B2) Suppose at least one of L1L2R3 or R1L2R3 $\in \text{supp}(\sigma_2)$. Then D1U2 and D1D2D3 $\notin \text{supp}(\sigma_1)$. This suggests that we can partition the set *B* into the sets *E* and *F*, where $B \supset E = \{$supports|at least one of L1L2R3, R1L2R3 $\in \text{supp}(\sigma_2)$, D1U2, D1D2D3 $\notin \text{supp}(\sigma_1)\,\}$ and $B \supset F = \{$supports| L1L2R3,R1L2R3 $\notin \text{supp}(\sigma_2)\,\}$

C) Consider the set of supports $A \supset C = \{$ supports|at least one of D1U2, D1D2U3, D1D2D3 $\in$ supp(σ_1) , R1L2L3 $\notin$ supp(σ_2)$\}$. The “Feng-Powell reduced form” of the game which does not include the strategies which are not in the support of one of the players is:

	R1L2R3		R1R2	
U1	5	5	5	5
D1U2	1	2	7	3
D1D2U3	1.5	4	7	3
D1D2D3	0.5	1	7	3

C1) I notice that D1U2 is weakly dominated by D1D2U3 and D1D2D3 is weakly dominated by D1D2U3.

C2) Suppose R1L2R3 $\in$ supp(σ_2) . Then D1U2 and D1D2D3 $\notin$ supp(σ_1) . This suggests that we can partition the set C into the sets G and H where $C \supset G = \{$ supports| R1L2R3 $\in$ supp(σ_2) , D1U2, D1D2D3 $\notin$ supp(σ_1) $\}$ and $C \supset H = \{$ supports| R1L2R3 $\notin$ supp(σ_2) $\}$.

D) Consider the set of supports $A \supset D = \{$ supports|D1U2 $\notin$ supp(σ_1) , D1D2U3 $\notin$ supp(σ_1) , D1D2D3 $\notin$ supp(σ_1) $\}$. The “Feng-Powell reduced form” of the game which does not include the strategies which are not in the support of one of the players is:

	R1L2L3		R1L2R3		R1R2	
U1	5	5	5	5	5	5

D4) This Feng-Powell reduced form game clearly gives rise to a continuum of mixed strategies for this game of the class:

$$\sigma_D = ((\text{U1}),(p_1 \text{R1L2L3} + p_2 \text{R1L2R3} + (1 - p_1 - p_2) \text{R1R2})).$$

E) Consider the set of supports $B \supset E = \{$ supports|at least one of L1L2R3, R1L2R3 $\in$ supp(σ_2) , D1U2, D1D2D3 $\notin$ supp (σ_1) $\}$. The “Feng-Powell reduced form” of the game which does not include the strategies which are not in the support of one of the players is:

	L1L2R3		L1R2		R1L2L3		R1L2R3		R1R2	
D1D2U3	1.5	4	7	3	2	2	1.5	4	7	3

E4) This Feng-Powell reduced form game clearly gives rise to a continuum of mixed strategies for this game of the classes:

$$\sigma_{E1} = ((\text{D1D2U3}),(p_1 \text{L1L2R3+} (1 - p_1) \text{R1L2R3}))$$

$$\sigma_{E2} = ((\text{D1D2U3}),(p_1 \text{L1R2+} (1 - p_1) \text{R1R2}))$$

F) Consider the set of supports $B \supset F = \{\,\text{supports}|\text{L1L2R3,R1L2R3} \notin \text{supp}\,(\sigma_2)\,\}$. The “Feng-Powell reduced form” of the game which does not include the strategies which are not in the support of one of the players is:

	L1R2		R1L2L3		R1R2	
D1U2	7	3	1	2	7	3
D1D2U3	7	3	2	2	7	3
D1D2D3	7	3	2	2	7	3

F1) I notice that R1L2L3 is strictly dominated by L1R2 and R1R2. This gives rise to the reduced form:

	L1R2		R1L2L3		R1R2	
D1U2	7	3			7	3
D1D2U3	7	3			7	3
D1D2D3	7	3			7	3

F4) This Feng-Powell reduced form game clearly gives rise to a continuum of mixed strategies for this game of the class:

$$\sigma_F = ((\,q_1\,\text{D1U2} + q_2\,\text{D1D2U3} + (1 - q_1 - q_2)\,\text{D1D2D3}),(\,p_1\,\text{L1R2} + (1 - p_1)\,\text{R1R2}))$$

G) Consider the set of supports $C \supset G = \{\,\text{supports}|\ \text{R1L2R3} \in \text{supp}(\sigma_2)\,,\ \text{D1U2, D1D2D3} \notin \text{supp}(\sigma_1)\,\}$. The “Feng-Powell reduced form” of the game which does not include the strategies which are not in the support of one of the players is:

	R1L2R3		R1R2	
U1	5	5	5	5
D1D2U3	1.5	4	7	3

G4) There are no mixed NE in this game. (I checked using the usual methods, which I will not detail here since it is boring.)

H) Consider the set of supports $C \supset H = \{\,\text{supports}|\ \text{R1L2R3} \notin \text{supp}(\sigma_2)\,\}$. The “Feng-Powell reduced form” of the game which does not include the strategies which are not in the support of one of the players is:

	R1R2	
U1	5	5
D1U2	7	3
D1D2U3	7	3
D1D2D3	7	3

H1) I notice that U1 is strictly dominated for I by D1U2, D1D2U3, and D1D2D3.

	R1R2	
U1	[redacted]	
D1U2	7	3
D1D2U3	7	3
D1D2D3	7	3

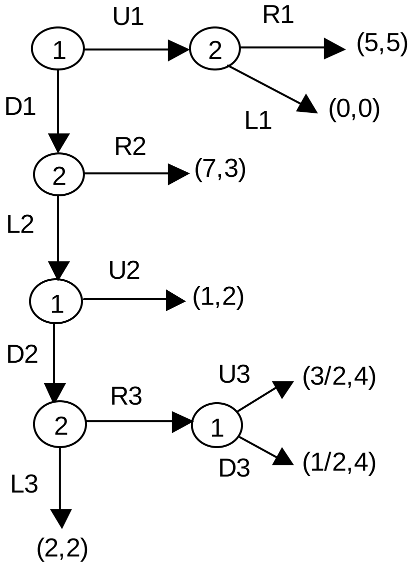
H4) This Feng-Powell reduced form game clearly gives rise to a continuum of mixed strategies for this game of the class:

$$\sigma_H = ((\, q_1\,\mathrm{D1U2} + q_2\,\mathrm{D1D2U3} + (1-q_1-q_2)\,\mathrm{D1D2D3}),(\mathrm{R1R2})).$$

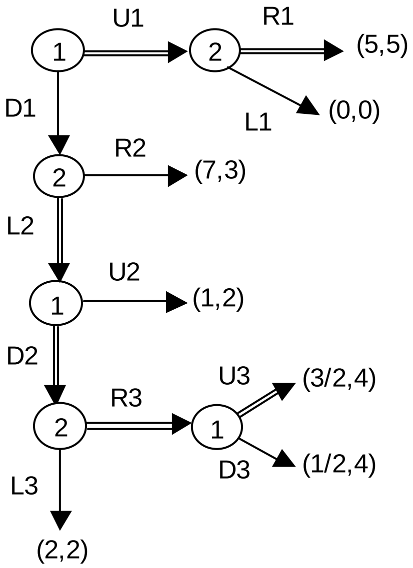
Therefore, all the mixed NE of this game are of the form: (Where the *p*s and the *q*s follow the usual rules for a probability:

$$\sigma_D = ((\mathrm{U1}),(\,p_1\,\mathrm{R1L2L3} + p_2\,\mathrm{R1L2R3} + (1-p_1-p_2)\,\mathrm{R1R2}))$$
$$\sigma_{E1} = ((\mathrm{D1D2U3}),(\,p_1\,\mathrm{L1L2R3} + (1-p_1)\,\mathrm{R1L2R3}))$$
$$\sigma_{E2} = ((\mathrm{D1D2U3}),(\,p_1\,\mathrm{L1R2} + (1-p_1)\,\mathrm{R1R2}))$$
$$\sigma_F = ((\,q_1\,\mathrm{D1U2} + q_2\,\mathrm{D1D2U3} + (1-q_1-q_2)\,\mathrm{D1D2D3}),(\,p_1\,\mathrm{L1R2} + (1-p_1)\,\mathrm{R1R2}))$$
$$\sigma_H = ((\,q_1\,\mathrm{D1U2} + q_2\,\mathrm{D1D2U3} + (1-q_1-q_2)\,\mathrm{D1D2D3}),(\mathrm{R1R2}))$$

3) Because we are using expected utility, this game is equivalent to the following game:



Performing the usual test for finding subgame perfect equilibrium, I find:



That is, (U1D2U3,R1L2R3) is subgame perfect.

Dominance and Nash Equilibrium

> Prove that a profile is a Nash equilibrium of a game if and only if it is the Nash equilibrium of the game in which strategies have been removed by iterated strict dominance. Prove that a Nash equilibrium of a game in which strategies have been removed by iterated weak dominance is a Nash equilibrium of the original game. Give an example of a Nash equilibrium of a game that is not a Nash equilibrium of the game where strategies have been removed by iterated weak dominance.

Before I begin these proofs, I start with several definitions and a couple lemmas, which will not be proven (they have been proven elsewhere):

Definition: A game *G* is an ordered triple (N, Σ, u) consisting of players, strategies, and payoffs for the players.

Disclaimer: I will ignore restrictions of the payoff function when I reduce the domain on which it is defined. It would become far too tedious, and ultimately nothing more than an exercise in mathematical accounting, to account for this rigorously.

Definition: A Nash equilibrium of *G* is a strategy profile σ^* satisfying $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*) \ \forall s_i' \in S_i$.

Lemma 1: σ^* is a Nash equilibrium of *G* iff it satisfies: $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \ \forall \sigma_i' \in \Sigma_i$.

Definition: The set of all mixed strategies which have the strategy s_i in their support is denoted: $\psi(s_i) \equiv \{\sigma_i \in \Sigma_i \mid s_i \in \operatorname{supp}(\sigma_i)\}$.

Definition: For some $j \in N$, a pure strategy $s_j{}^d \in S_j$ is called <u>dominated</u> if $\exists \sigma_j \in \Sigma_j \setminus \psi(s_j{}^d)$ satisfying $u_j(s_j{}^d, s_{-j}) < u_j(\sigma_j, s_{-j})$ $\forall s_{-j} \in S_{-j}$.

Lemma 2: $s_j{}^d \in S_j$ is dominated iff $\exists \sigma_j \in \Sigma_j$ satisfying $u_j(s_j{}^d, \sigma_{-j}) < u_j(\sigma_j, \sigma_{-j})$ $\forall \sigma_{-j} \in \Sigma_{-j}$.

Definition: For some $j \in N$, a pure strategy $s_j{}^d \in S_j$ is called <u>weakly dominated</u> if $\exists \sigma_j \in \Sigma_j \setminus \psi(s_j{}^d)$ satisfying $u_j(s_j{}^d, s_{-j}) \le u_j(\sigma_j, s_{-j})$ $\forall s_{-j} \in S_{-j}$ with the inequality strict for some $s_{-j}' \in S_{-j}$.

Lemma 3: $s_j{}^{wd} \in S_j$ is weakly dominated iff $\exists \sigma_j \in \Sigma_j$ satisfying $u_j(s_j{}^{wd}, \sigma_{-j}) \le u_j(\sigma_j, \sigma_{-j})$ $\forall \sigma_{-j} \in \Sigma_{-j}$ with the inequality strict for some $\sigma_j' \in \sigma$.

Proof 1: Prove that if a profile is a Nash equilibrium of a game, then it is a Nash equilibrium of the game in which strategies have been removed by iterated strict dominance

Suppose we have the game $G = (N, \Sigma, u)$. Suppose for G $\exists$ a dominated strategy for some $j \in N$: $s_j{}^d \in S_j$. (If no such strategy existed, then iterated strict dominance would be impossible and the proof would be trivially true.)

Claim: $\forall s_i \in \operatorname{supp}(\sigma_i{}^*)$, s_i is not dominated.

Proof of claim (by contradiction): Take $i \in N$ arbitrary and suppose

> $\exists\, s_i \in \operatorname{supp}(\sigma_i{}^*) \ni \exists \sigma_i \in \Sigma_i \setminus \psi(s_i)$ satisfying $u_i(s_i, s_{-i}) < u_i(\sigma_i, s_{-i})$ $\forall s_{-i} \in S_{-i}$. (i.e. s_i is dominated) Since this holds $\forall s_{-i} \in S_{-i}$, it holds for all mixed strategies $\sigma_{-i} \in \Sigma_{-i}$ by Lemma 2. In particular, we have: $u_i(s_i, \sigma_{-i}{}^*) < u_i(\sigma_i, \sigma_{-i}{}^*)$. Since $s_i \in \operatorname{supp}(\sigma_i{}^*)$, it follows that $u_i(s_i, \sigma_{-i}{}^*) = u_i(\sigma_i{}^*, \sigma_{-i}{}^*)$ since in a Nash equilibrium, each player is indifferent between all strategies on which he places positive probability. Therefore, $u_i(\sigma_i{}^*, \sigma_{-i}{}^*) = u_i(s_i, \sigma_{-i}{}^*) < u_i(\sigma_i, \sigma_{-i}{}^*)$ for some $\sigma_i \in \Sigma_i$, which contradicts $\sigma_i{}^*$ being a Nash equilibrium. Therefore, s_i is not dominated.

Construct the game $G' = (N, \Sigma \setminus \psi(s_j{}^d), u)$. G' is the game where for some particular $j \in N$, the strategy $s_j{}^d \in S_j$ has been eliminated by iterated strict dominance.

Since no $s_i \in \text{supp}(\sigma_i *)$ is dominated, no such s_i could have been eliminated by iterated strict dominance, so $\sigma^* \in \Sigma \setminus \psi(s_j^d)$. In addition, since $u_i(\sigma_i *, \sigma_{-i} *) \geq u_i(s_i ', \sigma_{-i} *)$ $\forall s_i ' \in S_i$, it follows that $u_i(\sigma_i *, \sigma_{-i} *) \geq u_i(s_i ', \sigma_{-i} *)$ $\forall s_i ' \in S_i \setminus s_i^d$ by the independence of irrelevant alternatives axiom (see Binmore). This proves that a Nash equilibrium of *G* is a Nash equilibrium of *G'*. (i.e. a Nash equilibrium of a game is a Nash equilibrium of a game where a single strategy has been removed by iterated strict dominance.) If *G'* contains no more dominated strategies, then we are done.

Otherwise, we can repeat the process again until we run out of dominated strategies. As proven, a Nash equilibrium of each game will be a Nash equilibrium of the game that follows after a single strategy has been removed by iterated strict dominance. Therefore, a Nash equilibrium of the original game will be a Nash equilibrium of the game where all dominated strategies have been removed by iterated strict dominance. Q.E.D.

Proof 2: Prove that if a profile is a Nash equilibrium of the game in which strategies have been removed by iterated strict dominance, then it is a Nash equilibrium of the original game.

It suffices to show that if $\sigma *$ Nash equilibrium of the game $G' = (N, \Sigma \setminus \psi(s_j^d), u')$ it is also a Nash equilibrium of the game $G = (N, \Sigma, u)$. Then an inductive argument can establish that $\sigma *$ is a Nash equilibrium of the original game.

Since for some $j \in N$, s_j^d was dominated in *G*, by definition, we have that $\exists \sigma_j \in \Sigma_j \setminus \psi(s_j^d) \ni u_j(s_j^d, s_{-j}) < u_j(\sigma_j, s_{-j})\ \forall s_{-j} \in S_{-j}$. Let $\sigma *$ be a Nash equilibrium of the game $G' = (N, \Sigma \setminus \psi(s_i^d), u')$. That is, $\forall i \in N \setminus j$, $u_i '(\sigma_i *, \sigma_{-i} *) \geq u_i '(s_i, \sigma_{-i} *)$ $\forall s_i \in S_i$. For j, $u_j '(\sigma_j *, \sigma_{-j} *) \geq u_j '(s_j, \sigma_{-j} *)\ \forall s_j \in S_j \setminus s_j^d$.

In order to establish that $\sigma *$ is a Nash equilibrium for *G*, we need to show that $\forall i \in N$, $u_i(\sigma_i *, \sigma_{-i} *) \geq u_i(s_i, \sigma_{-i} *)\ \forall s_i \in S_i$. Therefore, it suffices to show that $u_j(\sigma_j *, \sigma_{-j} *) \geq u_j(s_j^d, \sigma_{-j} *)$. The rest has already been established by the definition of $\sigma *$ being a Nash equilibrium of *G'*.

Suppose in order to get a contradiction, that $u_j(\sigma_j *, \sigma_{-j} *) < u_j(s_j^d, \sigma_{-j} *)$. (That is, that $\sigma *$ is not a Nash equilibrium of *G*)

Since s_j^d is a dominated strategy, $\exists \sigma_j \in \Sigma_j \setminus \psi(s_j^d)$ satisfying $u_j(s_j^d, s_{-j}) < u_j(\sigma_j, s_{-j})\ \forall s_{-j} \in S_{-j} \Leftrightarrow u_j(s_j^d, \sigma_{-j}) < u_j(\sigma_j, \sigma_{-j})\ \forall \sigma_j \in \Sigma_{-j}$ by Lemma 2. Since this holds $\forall \sigma_{-j} \in \Sigma_{-j}$, it holds for $\sigma_{-j} * \Rightarrow u_j(s_j^d, \sigma_{-j} *) < u_j(\sigma_j, \sigma_{-j} *)$.

Therefore, $u_j(\sigma_j *, \sigma_{-j} *) < u_j(s_j^d, \sigma_{-j} *) < u_j(\sigma_{j,} \sigma_{-j} *)$ for some $\sigma_j \in \Sigma_j \setminus \psi(s_j^d)$. That is, $\sigma *$ is not a NE of *G'* by Lemma 1, which is a contradiction. Therefore, $\sigma *$ must be a Nash equilibrium of *G*.

Without loss of generality, it suffices to assume that *G'* is the final game remaining after iterated strict dominance. Suppose *G* is the original game. Then we are done. If *G* is not the original game, we can iteratively add dominated strategies of the original game to our *G* to create new games (of which $\sigma *$ will be a Nash equilibrium) until we reach the original game. $\sigma *$ will be a Nash equilibrium of this game. Q.E.D.

Proof 3: Prove that if a profile is a Nash equilibrium of the game in which strategies have been removed by iterated weak dominance, then it is a Nash equilibrium of the original game.

The proof here is almost identical. It suffices to show that if $\sigma *$ Nash equilibrium of the game $G' = (N, \Sigma \setminus \psi(s_j^{wd}), u')$ it is also a Nash equilibrium of the game $G = (N, \Sigma, u)$. Then an inductive argument can establish that $\sigma *$ is a Nash equilibrium of the original game.

Since for some $j \in N$, s_j^{wd} was weakly dominated in *G*, by definition, we have that $\exists \sigma_j \in \Sigma_j \setminus \psi(s_j^{wd}) \ni u_j(s_j^{wd}, s_{-j}) \leq u_j(\sigma_j, s_{-j})\ \forall s_{-j} \in S_{-j}$ with at least one strict. Let $\sigma *$ be a Nash equilibrium of the game $G' = (N, \Sigma \setminus \psi(s_i^{wd}), u')$. That is, $\forall i \in N \setminus j$, $u_i'(\sigma_i *, \sigma_{-i} *) \geq u_i'(s_i, \sigma_{-i} *) \quad \forall s_i \in S_i$. For j, $u_j'(\sigma_j *, \sigma_{-j} *) \geq u_j'(s_j, \sigma_{-j} *)$ $\forall s_j \in S_j \setminus s_j^{wd}$.

In order to establish that $\sigma *$ is a Nash equilibrium for *G*, we need to show that $\forall i \in N$, $u_i(\sigma_i *, \sigma_{-i} *) \geq u_i(s_i, \sigma_{-i} *)\ \forall s_i \in S_i$. Therefore, it suffices to show that $u_j(\sigma_j *, \sigma_{-j} *) \geq u_j(s_j^{wd}, \sigma_{-j} *)$. The rest has already been established by the definition of $\sigma *$ being a Nash equilibrium of *G'*.

Suppose in order to get a contradiction, that $u_j(\sigma_j *, \sigma_{-j} *) < u_j(s_j^{wd}, \sigma_{-j} *)$. (That is, that $\sigma *$ is not a Nash equilibrium of *G*)

Since s_j^{wd} is a weakly dominated strategy, $\exists \sigma_j \in \Sigma_j \setminus \psi(s_j^{wd})$ satisfying $u_j(s_j^{wd}, s_{-j}) \leq u_j(\sigma_j, s_{-j})\ \forall s_{-j} \in S_{-j}$ with at least one strict $\Leftrightarrow$ $u_j(s_j^{wd}, \sigma_{-j}) \leq u_j(\sigma_j, \sigma_{-j}) \quad \forall \sigma_j \in \Sigma_{-j}$ with at least one strict by Lemma 2. Since this holds $\forall \sigma_{-j} \in \Sigma_{-j}$, it holds for $\sigma_{-j} * \Rightarrow$ $u_j(s_j^{wd}, \sigma_{-j} *) \leq u_j(\sigma_j, \sigma_{-j} *)$.

Therefore, $u_j(\sigma_j *, \sigma_{-j} *) < u_j(s_j^{wd}, \sigma_{-j} *) \leq u_j(\sigma_{j,} \sigma_{-j} *)$ for some $\sigma_j \in \Sigma_j \setminus \psi(s_j^{wd})$. That is, $\sigma *$ is not a NE of *G'* by Lemma 1, which is a contradiction. Therefore, $\sigma *$ must be a Nash equilibrium of *G*.

Without loss of generality, it suffices to assume that *G'* is the final game remaining after iterated weak dominance. Suppose *G* is the original game. Then we are done. If *G* is not the original game, we can iteratively add weakly dominated strategies of the original game to our *G* to create new games (of which $\sigma *$ will be a Nash equilibrium) until we reach the original game. $\sigma *$ will be a Nash equilibrium of this game. Q.E.D.

Example of a Nash equilibrium of a game that is not a Nash equilibrium of the game where strategies have been removed by iterated weak dominance:

	L	R
U	3 2	2 3
D	3 2	2 2

⇒

	L	R
U	3 2	2 3
D	3 2	2 2

The pure strategy Nash equilibria are: $(U,R), (D,L), (D,R)$. If we perform iterated weak dominance on this game, the strategy L for player II is eliminated:

	L	R
U	[redacted]	2 3
D	[redacted]	2 2

However, this iteration removes the Nash equilibrium (D,L).

Backward Induction

> There are five pirates with names 1,2,3,4,5. They have just seized a hundred gold coins, and now it's time to share the loot. The bargaining rules are: Whoever has the lowest number as a name must propose a division of the one hundred coins to the remaining pirates. If the majority accepts the proposal, then the coins are allocated and the game ends. If the majority does not accept, then the proposer gets thrown overboard and the game is repeated with one less pirate. What should the first pirate propose?

The answer to this question clearly depends on how we define a majority and whether or not people get less utility from being thrown overboard than from acquiring zero coins. I will define a majority to be one half of the number of pirates voting plus one. (i.e. if there are four pirates, three pirates forms a majority) In addition, I will assume that the pirates prefer zero coins to being thrown overboard. If a pirate is indifferent between voting yes and voting no, he will vote no (I am throwing away some subgame perfect equilibria here, but this is just for brevity's sake).

Suppose only pirate 5 is left (1 pirate is needed for a majority): Pirate 5 will keep 100 coins for himself. (Pirate 5 will vote yes)

	Pirate 1	Pirate 2	Pirate 3	Pirate 4	Pirate 5
Coins	Overboard	Overboard	Overboard	Overboard	100

Suppose pirates 4 and 5 are left (2 pirates are needed for a majority): Since, if pirate 4 is thrown overboard, pirate 5 gets to keep 100 coins, pirate 5 will not accept any fewer than 101 coins. Pirate 4 cannot win a majority without pirate 5's approval. Thus, pirate 4 will be thrown overboard. (Pirate 4 will vote yes, pirate 5 will vote no)

	Pirate 1	Pirate 2	Pirate 3	Pirate 4	Pirate 5
Coins	Overboard	Overboard	Overboard	Overboard	100

Suppose pirates 3, 4, and 5 are left (2 pirates are needed for a majority): If a majority is not reached here, the results of the previous case will hold. That is, pirate 4 will get thrown overboard and pirate 5 will get 100 coins. Therefore, pirate 4 will accept any offer from pirate 3. Pirate 3 will then offer 0 coins to pirate 4, zero to pirate 5 and keep the rest for himself. (Pirate 3 will vote yes, pirate 4 will vote yes, pirate 5 will vote no)

	Pirate 1	Pirate 2	Pirate 3	Pirate 4	Pirate 5
Coins	Overboard	Overboard	100	0	0

Suppose pirates 2, 3, 4, and 5 are left (3 pirates are needed for a majority): Pirate 5 will accept anything greater than 0 coins, pirate 4 will accept anything greater than 0 coins, pirate 3 will not accept anything less than 101 coins. Thus, pirate 2 will offer 1 coin to pirate 4 and 1 coin to pirate 5, keeping the rest for himself. (Pirate 2 will vote yes, pirate 3 will vote no, pirate 4 will vote yes, pirate 5 will vote yes)

	Pirate 1	Pirate 2	Pirate 3	Pirate 4	Pirate 5
Coins	Overboard	98	0	1	1

Suppose all pirates are present (3 pirates are needed for a majority): Pirate 5 will accept anything greater than 1 coin, pirate 4 will accept anything greater than 1 coin, pirate 3 will accept anything greater than 0 coins, and pirate 2 will reject anything less than 99 coins.

Case 1: Pirate 1 will offer 1 coin to pirate 3, 2 coins to pirate 4, and keep the rest. (Pirate 1 will vote yes, pirate 2 will vote no, pirate 3 will vote yes, pirate 4 will vote yes, pirate 5 will vote no)

Case 2: Pirate 1 will offer 1 coin to pirate 3, 2 coins to pirate 5, and keep the rest. (Pirate 1 will vote yes, pirate 2 will vote no, pirate 3 will vote yes, pirate 4 will vote no, pirate 5 will vote yes)

Case 1	Pirate 1	Pirate 2	Pirate 3	Pirate 4	Pirate 5
Coins	97	0	1	2	0

Case 2	Pirate 1	Pirate 2	Pirate 3	Pirate 4	Pirate 5
Coins	97	0	1	0	2

Correlated Equilibrium

Consider the game

	L	R
U	0 0	2 1
D	1 2	0 0

Show that the correlated strategy profile

	L	R
U	1/3	1/3
D	1/3	0

is in fact a correlated equilibrium

Solution: It suffices to show that neither player has any incentive to deviate from the action which they are told to play.

Suppose player I is told to play U:

Player I knows that the randomizer has told player II to play L with probability .5:

L	R
1/2	1/2

$$\left.\begin{aligned} u_1(U) &= \left(\frac{1}{2}\right)(0) + \left(\frac{1}{2}\right)(2) = 1 \\ u_1(D) &= \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(0) = \frac{1}{2} \end{aligned}\right\} \text{Player I will pick U.}$$

Suppose player I is told to play D:

Player I knows that the randomizer has told player II to play L with probability 1:

$$\left.\begin{aligned} u_1(U) &= (1)(0) + (0)(2) = 0 \\ u_1(D) &= (1)(1) + (0)(0) = 1 \end{aligned}\right\} \text{Player I will pick D.}$$

Suppose player II is told to play L:

Player II knows that the randomizer has told player I to play U with probability .5:

U	1/2
D	1/2

$$\left.\begin{aligned} u_2(L) &= \left(\frac{1}{2}\right)(0)+\left(\frac{1}{2}\right)(2)=1 \\ u_2(R) &= \left(\frac{1}{2}\right)(1)+\left(\frac{1}{2}\right)(0)=\frac{1}{2} \end{aligned}\right\}\text{Player II will pick L.}$$

Suppose player II is told to play R:

Player II knows that the randomizer has told player I to play U with probability 1:

$$\left.\begin{aligned} u_2(L) &= (1)(0)+(0)(2)=0 \\ u_2(R) &= (1)(1)+(0)(0)=1 \end{aligned}\right\}\text{Player II will pick R.}$$

Thus, no matter what the randomizer tells the players to do, they will not deviate. That is, this is a correlated equilibrium.