9.4.9 Let f be a continuously differentiable function on $[-\pi, \pi]$ such that $\int_{-\pi}^{\pi} f(x)dx = 0$. Prove that

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx \geq \int_{-\pi}^{\pi} |f(x)|^2 dx$$

Prove that strict equality holds if and only if $f(x) = a\cos x + b\sin x$ for some constants a and b. Hint: use Parseval's relation.

Proof of exercise 9.4.9:

Since f is continuously differentiable on $[-\pi, \pi]$, its Fourier series converges uniformly to it by theorem 9.4.4. Thus, we have:

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} dx$.

Let $n = 0 \Rightarrow c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ by assumption.

$\Rightarrow f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{inx}$. Since the convergence is uniform and f is continuously differentiable, we have that $f'(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} inc_n e^{inx}$ by theorem 6.3.3.

By Parseval's relation, $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |c_n|^2$. In addition, we have that

$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^2 |c_n|^2$. Since $n \neq 0$, $|n| \geq 1$.

$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx = \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty} |n|^2 |c_n|^2 \geq \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$

Therefore, $\int_{-\pi}^{\pi} |f'(x)|^2 dx \geq \int_{-\pi}^{\pi} |f(x)|^2 dx$.

This result stems from the fact that $\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^2 |c_n|^2 \geq \sum |c_n|^2$. Suppose there exists some n' such that $|n'| \geq 2$ and $c_{n'} \neq 0$. Then $|n'|^2 |c_{n'}|^2 > |c_{n'}|^2$ and there is strict inequality. Thus, equality holds if and only if for all n such that $|n| \geq 2$, $c_n = 0$. Consider this case: (Note that c_1 and c_{-1} can be zero or nonzero)

$f(x) = c_1 e^{ix} + c_{-1}e^{-ix}$. By Euler's formula, we have:

$f(x) = c_1(\cos x + i\sin x) + c_{-1}(\cos x - i\sin x) = (c_1 + c_{-1})\cos x + i(c_1 - c_{-1})\sin x$.

That is, $f(x) = a\cos x + b\sin x$ where $a = c_1 + c_{-1}$ and $b = i(c_1 - c_{-1})$. Q.E.D.