

9.2.10 Suppose that the Fourier coefficients of a piecewise continuous function  $f$  satisfy  $\sum m^k |a_m| < \infty$  and  $\sum m^k |b_m| < \infty$  for some fixed positive integer  $k$  and that the Fourier series converges to  $f(x)$  for all  $x$ . Prove that  $f$  is  $k$  times continuously differentiable.

Proof of exercise 9.2.10:

We know that  $f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx$  since we assumed that the Fourier series converges to  $f(x)$  for all  $x \in \mathbf{R}$ .

Consider the sum of the derivatives of  $f_n$ :

$$\begin{aligned} \sum_{m=1}^{\infty} (-ma_m \sin mx + mb_m \cos mx) &\leq \sum_{m=1}^{\infty} m|a_m| |\sin mx| + m|b_m| |\cos mx| \\ &\leq \sum_{m=1}^{\infty} m|a_m| + m|b_m| = \sum_{m=1}^{\infty} m|a_m| + \sum_{m=1}^{\infty} m|b_m| < +\infty. \end{aligned}$$

That is, the sum of the derivatives converges uniformly by the Weierstrass M-Test. Therefore,  $f$  is continuously differentiable and  $f'(x) = \sum_{m=1}^{\infty} (-ma_m \sin mx + mb_m \cos mx)$  by theorem 6.3.3.

Let  $k$  be arbitrary but fixed. Consider the sum of the  $k$ -th derivatives of  $f_n$ :

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{d^k}{dx^k} (a_m \cos mx + b_m \sin mx) &= \sum_{m=1}^{\infty} \left( a_m \frac{d^k}{dx^k} \cos mx + b_m \frac{d^k}{dx^k} \sin mx \right) \\ &\leq \sum_{m=1}^{\infty} (m^k |a_m| + m^k |b_m|) \end{aligned}$$

since for each  $k$ ,  $\frac{d^k}{dx^k} \cos mx \leq m^k$  and  $\frac{d^k}{dx^k} \sin mx \leq m^k$ .

$$\sum_{m=1}^{\infty} (m^k |a_m| + m^k |b_m|) \leq \sum_{m=1}^{\infty} m^k |a_m| + \sum_{m=1}^{\infty} m^k |b_m| < +\infty.$$

That is, the sum of the  $k$ -th derivatives converges uniformly by the Weierstrass M-Test. Therefore,  $f$  is  $k$  times continuously differentiable and  $f^{(k)}(x) = \sum_{m=1}^{\infty} \frac{d^k}{dx^k} (a_m \cos mx + b_m \sin mx)$  by theorem 6.3.3 applied  $k$  times. Q.E.D.