6.5.7 Suppose that $z_n \to z$. Prove that

6.5.7.a $|z_n| \to |z|$

Lemma 6.5.7.a:

If $z, z' \in \mathbf{C}$, then $\left||z| - |z'|\right| \le |z - z'|$.

Proof of lemma 6.5.7.a:

$|z| = |z - z' + z'| \le |z - z'| + |z'|$ Þ by the triangle inequality of the modulus.

$|z'| = |z' - z + z| \le |z' - z| + |z| \Rightarrow |z'| - |z| \le |z' - z| = |z - z'|$ by the triangle inequality of the modulus and homogeneity.

$\Rightarrow \left||z| - |z'|\right| = \max\{|z| - |z'|, |z'| - |z|\} \le |z - z'|$. Q.E.D.

Proof of exercise 6.5.7.a:

Since $z_n \to z$, we have that $\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \ge N, |z_n - z| < \varepsilon$. Let $\varepsilon > 0$ be arbitrary. Then $\forall n \ge N$, by lemma 6.5.7.a, we have that $\left||z_n| - |z|\right| \le |z_n - z| < \varepsilon$. That is, $|z_n| \to |z|$ as $n \to \infty$. Q.E.D.

6.5.7.b $\bar{z}_n \to \bar{z}$

Lemma 6.5.7.b

Let $z_n = x_n + iy_n$, $z = x + iy$. Then $z_n \to z \Rightarrow x_n \to x$ and $y_n \to y$.

Proof of lemma 6.5.7.b

$z_n \to z \Rightarrow \forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \ge N, |z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon$. Let $\varepsilon > 0$ be arbitrary. Then $\forall n \ge N$, $|x_n - x| \le \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon$ and $|y_n - y| \le \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon$. That is, $x_n \to x$ and $y_n \to y$. Q.E.D.

Proof of exercise 6.5.7.b:

Since $z_n \to z \Rightarrow x_n \to x$ and $y_n \to y$, if we can show that $x_n \to x$ and $y_n \to y$ $\Rightarrow \bar{z}_n \to \bar{z}$ if, then we are done.

Suppose $x_n \to x$ and $y_n \to y$. Then

$\forall \varepsilon > 0\ \exists N_1(\varepsilon) \in \mathbf{N} \ni \forall n \ge N_1, |x_n - x| < \frac{\varepsilon}{\sqrt{2}}$ and $\exists N_2(\varepsilon) \in \mathbf{N} \ni \forall n \ge N_2$,

$|y_n - y| < \frac{\varepsilon}{\sqrt{2}}$.

Let $\varepsilon > 0$ be arbitrary. Then $\forall n \ge \max\{N_1, N_2\}$, we have that

$|\bar{z}_n - \bar{z}| = |x_n - iy_n - x + y| = |x_n - x + i(y - y_n)| = \sqrt{(x_n - x)^2 + (y - y_n)^2}$

$$< \sqrt{\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2} = \sqrt{2\frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon.$$ That is, $\bar{z}_n \to \bar{z}$.

Thus, we have $z_n \to z \Rightarrow x_n \to x \wedge y_n \to y \Rightarrow \bar{z}_n \to \bar{z}$. Q.E.D.

6.5.7.c $\{z_n\}$ is a bounded sequence.
Proof of exercise 6.5.7.c:
For this exercise, I will assume that a sequence of complex numbers is "bounded" if $\exists M > 0 \ni \forall n, |z_n| \le M$. I saw no formal definition in the text.

Since $z_n \to z$ as $n \to \infty$, we have that $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \ge N, |z_n - z| < \varepsilon$. By lemma 6.5.7.a, we also have that $|z_n| - |z| \le |z_n - z| < \varepsilon \Rightarrow |z_n| < |z| + \varepsilon$. Let $\varepsilon = 1$. Then $\forall n \ge N(1)$, we have that $|z_n| < |z| + 1$. Let $M = \{|z_1|, \ldots, |z_{N-1}|, |z| + 1\}$. Then $\forall n, |z_n| \le M$. *i.e.* $\{z_n\}$ is a bounded sequence. Q.E.D.