6.4.9 Is there a power series which converges to the function $f(x) = |x|$ for all x?

Lemma 6.4.9.a:

$|x|$ is not differentiable at $x = 0$.

Proof of lemma 6.4.9.a:

Consider the difference quotient $\dfrac{|x+h| - |x|}{h}$.

$$\lim_{h \to 0^-} \frac{|x+h| - |x|}{h} = \lim_{h \to 0^-} \frac{-x-h+x}{h} = -1$$

$$\lim_{h \to 0^+} \frac{|x+h| - |x|}{h} = \lim_{h \to 0^+} \frac{x+h-x}{h} = 1$$

Therefore, $\lim_{h \to 0} \dfrac{|x+h| - |x|}{h}$ does not exist and $|x|$ is not differentiable at $x = 0$.

Answer to exercise 6.4.9:

Let $x_0 \in \mathbf{R}$ be arbitrary. Suppose $\sum_{n=0}^{\infty} a_n (x - x_0)^n = |x|$ for all $x \in \mathbf{R}$. Since this holds for all $x \in \mathbf{R}$, this implies that the radius of convergence for $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is $R = \infty$.

By theorem 6.4.2, it then follows that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is infinitely often continuously differentiable on any subdisk of its radius of convergence. In particular, $\sum_{n=0}^{\infty} a_n (x - x_0)^n = |x|$ is continuously differentiable on the subdisk $(-\varepsilon, 2x_0 + \varepsilon) \Rightarrow |x|$ is continuously differentiable at $x = 0$, which is a contradiction, by lemma 6.4.9.a. Therefore, no such power series exists. Q.E.D.