6.4.11 Find the radius of convergence of the series $\sum_{j=0}^{\infty}(j+1)(j+2)x^j$. Find the function to which the series converges.

Answer to exercise 6.4.11:

$a_j = (j+1)(j+2)$. $\alpha = \limsup |a_j|^{1/j} = 1$. Therefore, $R = 1$

Working backwards, consider the function $f(x) = -\frac{2}{(x-1)^3} \Rightarrow f(0) = 2$

$$f'(x) = \frac{6}{(x-1)^4} \qquad \Rightarrow f'(0) = 6$$

$$f''(x) = -\frac{24}{(x-1)^5} \qquad \Rightarrow f''(0) = 24$$

$$f'''(x) = \frac{120}{(x-1)^6} \qquad \Rightarrow f'''(0) = 120$$

$\vdots$

$$f^{(k)}(x) = (-1)^{k-1}\frac{(k+2)!}{(x-1)^{k+3}} \qquad \Rightarrow f^{(k)}(0) = (k+2)!$$

Then, $T^{(n)}(x,0) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x)^n$

$\Rightarrow T^{(n)}(x,0) = 2! + 3!x + \frac{4!}{2!}x^2 + \frac{5!}{3!}x^3 + \ldots + \frac{(n+2)!}{n!}x^n$

$\Rightarrow T^{(n)}(x,0) = (2)(1) + (3)(2)x + (4)(3)x^2 + (5)(4)x^3 + \ldots + (n+2)(n+1)x^n$

For $x \in (-1,1)$, we then have that $f(x) = \sum_{j=0}^{\infty}(j+1)(j+2)x^j$.