6.3.12 Prove that

$$\zeta(x) \equiv \sum_{j=1}^{\infty} \frac{1}{j^x}$$

defines a continuous function for $x > 1$. The function $\zeta(x)$ is called the Riemann zeta function.

Lemma 6.3.12:

$$(1,\infty) = \bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},\infty\right).$$

Proof of lemma 6.3.12:

In order to get a contradiction, suppose that for some $x \in (1,\infty)$, $x \notin \bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},\infty\right)$.

Since $x \in (1,\infty)$, $x > 1$ so $\exists\, c > 0 \ni x = 1 + c$. Take $n^* \geq \frac{1}{c}$. Then $1+\frac{1}{n} \leq 1 + c = x$ $\Rightarrow x \in \left[1+\frac{1}{n^*},\infty\right)$, which is a contradiction. Therefore, $x \in \bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},\infty\right)$ $\Rightarrow (1,\infty) \subset \bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},\infty\right)$

Let $x \in \bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},\infty\right) \Rightarrow$ For some $N \in \mathbf{N}$, $x \in \left[1+\frac{1}{N},\infty\right) \subset (1,\infty)$. Therefore, $x \in (1,\infty)$ $\Rightarrow \bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},\infty\right) \subset (1,\infty)$

Therefore, $(1,\infty) = \bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},\infty\right)$ Q.E.D.

Proof of exercise 6.3.12:

For $j = 1$, $f_j(x) = 1\ \forall x > 1$.

For $j > 1$, $f_j(x) = \frac{1}{j^x}$ is strictly decreasing in x for $x > 1$ (since $f_j'(x) = \frac{-\ln j}{j^x} < 0\ \forall x > 1,\ j > 1$). Consider the sequence of half-closed intervals $\left[1+\frac{1}{n},\infty\right)$. Suppose $x \in \left[1+\frac{1}{n},\infty\right)$. For each j, for each n, $f_j(x)$ is bounded from above by $f_j\left(1+\frac{1}{n}\right) = \frac{1}{j^{1+1/n}} \equiv M_j^{(n)}$ (since $f_j(x)$ is strictly decreasing in x).

Since $(1+1/n)>1$ for each n, it follows that $\sum_{j=1}^{\infty} M_j^{(n)}$ is a convergent *p*-series. Therefore, by the Weierstrass M-Test, since for all j, $f_j(x)=\frac{1}{j^x}$ is continuous on $(1,\infty)$, it follows that $\zeta^{(k)}(x)=\sum_{j=1}^{\infty}\frac{1}{j^x}$, where $\zeta^{(k)}(x)=\zeta(x)$ restricted to the domain $\bigcup_{n=1}^{k}\left[1+\frac{1}{n},\infty\right)$, is continuous on $\bigcup_{n=1}^{k}\left[1+\frac{1}{n},\infty\right)$ for each $k\in\mathbf{N}$. Taking the limit as $k\to\infty$, by lemma 6.3.12, $\zeta(x)$ is continuous on $(1,\infty)$. Q.E.D.