6.2.1 Determine whether the following series converge or diverge:

6.2.1.a $\sum \frac{3^j}{j!}$

Answer to exercise 6.2.1.a:

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right| = \left|\frac{3 \cdot 3^n}{(n+1)n!} \cdot \frac{n!}{3^n}\right| = \left|\frac{3}{n+1}\right|$$

$\limsup \frac{|a_{n+1}|}{|a_n|} = 0 < 1$. Therefore, by theorem 6.2.4.a (The Ratio Test), $\sum \frac{3^j}{j!}$ converges absolutely. Thus, by theorem 6.2.1.c, $\sum \frac{3^j}{j!}$ converges

6.2.1.b $\sum \frac{\sqrt{n}}{1+n^2}$

Answer to exercise 6.2.1.b:

$\sum \frac{\sqrt{n}}{1+n^2} \le \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$ which converges by the *p*-test. Therefore, by theorem 6.2.2.a, $\sum \frac{\sqrt{n}}{1+n^2}$ converges.

6.2.1.c $\sum \frac{2^j}{j^2}$

Answer to exercise 6.2.1.c:

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n}\right| = \left|\frac{2 \cdot 2^n}{(n+1)^2} \cdot \frac{n^2}{2^n}\right| = \left|\frac{2n^2}{(n+1)^2}\right|$$

$\limsup \frac{|a_{n+1}|}{|a_n|} = 2 > 1$. Therefore, by theorem 6.2.4.a (The Ratio Test), $\sum \frac{2^j}{j^2}$ diverges.

6.2.1.d $\sum \frac{j^2}{j!}$

Answer to exercise 6.2.1.d:

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2}\right| = \left|\frac{(n+1)^2}{(n+1)n!} \cdot \frac{n!}{n^2}\right| = \left|\frac{n+1}{n^2}\right|$$

$\limsup \frac{|a_{n+1}|}{|a_n|} = 0 < 1$. Therefore, by theorem 6.2.4.a (The Ratio Test), $\sum \frac{j^2}{j!}$ converges absolutely. Thus, by theorem 6.2.1.c, $\sum \frac{j^2}{j!}$ converges

6.2.1.e $\sum\left(\sqrt{j+1}-\sqrt{j}\right)$

Answer to exercise 6.2.1.e:

$\sum\left(\sqrt{j+1}-\sqrt{j}\right)=\sum\left(\frac{j+1-j}{\sqrt{j+1}+\sqrt{j}}\right)=\sum\left(\frac{1}{\sqrt{j+1}+\sqrt{j}}\right)\geq\sum\frac{1}{2\sqrt{j+1}}$ which diverges by the p-test. Therefore, by theorem 6.2.2.b, $\sum\left(\sqrt{j+1}-\sqrt{j}\right)$ diverges.

6.2.1.f $\sum\frac{\sqrt{j+1}-\sqrt{j}}{j}$

Answer to exercise 6.2.1.f:

$\sum\frac{\sqrt{j+1}-\sqrt{j}}{j}=\sum\frac{j+1-j}{j\left(\sqrt{j+1}+\sqrt{j}\right)}\leq\sum\frac{1}{j\left(2\sqrt{j}\right)}=\sum\frac{1}{2j^{3/2}}$ which converges by the p-test. Therefore, by theorem 6.2.2.a, $\sum\frac{\sqrt{j+1}-\sqrt{j}}{j}$ converges.

6.2.1.g $\sum e^{-j+\sin j}$

Answer to exercise 6.2.1.g:

$\sum e^{-j+\sin j}\leq\sum e^{-j+1}=e\sum e^{-j}=e\sum b_j$. $\left|b_j\right|^{1/j}=\left|e^{-j}\right|^{1/j}=\left|e^{-1}\right|$ $\limsup\left|e^{-1}\right|=e^{-1}<1$. Therefore, $\sum b_j$ converges by theorem 6.2.3 (The Root Test). Therefore, by theorem 6.2.2.a, $\sum e^{-j+\sin j}$ converges.

6.2.1.h $\sum\frac{j+\cos j}{j}$

Answer to exercise 6.2.1.h:

$a_j=\frac{j+\cos j}{j}=1+\frac{\cos j}{j}\to 1$ as $j\to\infty$. Therefore, by the contrapositive of theorem 6.2.1.b, $\sum\frac{j+\cos j}{j}$ does not converge.