5.8.9 Recall from linear algebra that a set of vectors $\{v_i\}_{i=1}^{m}$ is said to be linearly independent if no linear combination $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_j v_n$ is the zero vector unless $\alpha_i = 0$ for all *i*. A vector space *V* is said to have dimension N if every set of N independent vectors $\{v_i\}_{i=1}^{N}$ spans *V*; that is, every vector in *V* can be written as a linear combination of the v_i. If *V* has dimension N for some N, *V* is said to be finite dimensional.

5.8.9.a Show that $\mathbf{R}^n$ has dimension *n*.
Proof of exercise 5.8.9.a:

Take any arbitrary vector $x \in \mathbf{R}^n$. $x = (x_1, x_2, ..., x_n)$ where $x_i \in \mathbf{R}\ \forall i$.
Consider the vectors $e_1 = (1,0,...,0), e_2 = (0,1,0,...,0),..., e_n = (0,0,...,0,n)$
$e_i \in \mathbf{R}^n\ \forall i$.
Then $x = x_1 e_1 + x_2 e_2 + ... + x_n e_n$. i.e. Any vector in $\mathbf{R}^n$ can be written as a linear combination of the e_i's. Therefore, $\{e_i \mid i = 1,...,n\}$ is a basis for $\mathbf{R}^n$. Since $Card\{e_i \mid i = 1,...,n\} = n$, we can conclude that $\mathbf{R}^n$ has dimension *n*. Q.E.D.

5.8.9.b Show that $\{1, x, x^2, ..., x^n\}$ is an independent set of vectors in $C[a,b]$ for each *n*.
Proof of exercise 5.8.9.b:

Take $n \in \mathbf{N}$ arbitrary.
Consider the polynomial $\alpha_0 1 + \alpha_1 x + \alpha_2 x^2 + ... + \alpha_n x^n \in C[a,b]$.
$0 \in C[a,b]$ is the function $g \in C[a,b] \ni g(x) = 0\ \forall x \in [a,b]$.
$\alpha_0 1 + \alpha_1 x + \alpha_2 x^2 + ... + \alpha_n x^n = 0$ The only way for a polynomial of degree *n* to be equal to the zero function is if its coefficients $\alpha_i = 0\ \ \forall i \in \{1,...,n\}$.
That is, $a_0 1 + \alpha_1 x + ... + \alpha_n x^n = 0$ iff $\alpha_0 = \alpha_1 = ... = \alpha_n = 0$. i.e $\{1, x, ..., x^n\}$ is a linearly independent set of vectors in $C[a,b]$. Since n was arbitrary, we can conclude that this holds for each *n*. Q.E.D.

5.8.9.c Prove that $C[a,b]$ is not finite dimensional.
Proof of exercise 5.8.9.b:

In order to get a contradiction, suppose that $C[a,b]$ is finite dimensional.
Assume $C[a,b]$ has dimension *N*.
Since $\{1, x, x^2, ..., x^n\}$ is a linearly independent set of vectors in $C[a,b]$ for each *n* from part b, we have that, given any $N \in \mathbf{N}$, $\{1, x, x^2, ..., x^N\}$ is a linearly independent set of vectors in $C[a,b]$, but we also have that $\{1, x, x^2, ..., x^N, x^{N+1}\}$ is linearly independent. Therefore, $C[a,b]$ has dimension of at least $N+1$, which is a contradiction. $\rightarrow\leftarrow$ Therefore, $C[a,b]$ is not finite dimensional.
Q.E.D.