5.8.7 Show that

$$\|f\| = \int_{-\infty}^{\infty} |f(x)| e^{-x^2}\, dx$$

is a norm on the space of bounded continuous functions on $\mathbf{R}$.

A norm satisfies the following three properties:

i) $\|f\| = 0$ if f is identically 0 and $\|f\| > 0$ if f is not identically 0.

ii) $\forall \alpha \in \mathbf{R}, \|\alpha f\| = |\alpha| \|f\|.$

iii) $\forall f, g \in V, \|f + g\| \le \|f\| + \|g\|.$

Proof of exercise 5.8.7:

i) Suppose $f = 0\ \forall x \in \mathbf{R}$. Then $|f(x)| = 0\ \forall x \in \mathbf{R}$.

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)| e^{-x^2}\, dx = \int_{-\infty}^{\infty} 0\ dx = 0$$

Suppose $\exists\, c \in \mathbf{R} \ni f(c) \ne 0$. Then $|f(c)| = d > 0$. Since f is continuous, $|f|$ is continuous.

$\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0 \ni \forall c, c'$ satisfying $|c - c'| < \delta,$ we have $|f(c) - f(c')| < \varepsilon.$ Since this holds $\forall \varepsilon > 0,$ it holds for $\varepsilon = \frac{d}{2}$. That is, $\exists \delta > 0 \ni \forall c' \in (c - \delta, c + \delta),$

$$|f(c)| - |f(c')| \le |f(c) - f(c')| < \frac{d}{2} \Rightarrow 0 < d - \frac{d}{2} = \frac{d}{2} < |f(c')|$$

We also have that on the interval $(c - \delta, c + \delta),\ e^{-x^2} \ge \min\{e^{-(c-\delta)^2}, e^{-(c+\delta)^2}\} \equiv \tau$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)| e^{-x^2}\, dx \ge \int_{c-\delta}^{c+\delta} |f(x)| e^{-x^2}\, dx$$

Since $|f(x)| e^{-x^2} \ge |f(x)|\tau\ \forall x \in (c - \delta, c + \delta),$ we have by theorem 3.3.4,

$$\int_{c-\delta}^{c+\delta} |f(x)| e^{-x^2}\, dx \ge \int_{c-\delta}^{c+\delta} |f(x)| \tau dx \ge \int_{c-\delta}^{c+\delta} \frac{d}{2} \tau dx = \delta d \tau \equiv \varepsilon' > 0$$

Therefore, $\|f\| > 0$ if f is not identically 0.

ii) Let $\alpha \in \mathbf{R}$. Then $\|\alpha f\| = \int_{-\infty}^{\infty} |\alpha f(x)| e^{-x^2}\, dx = \int_{-\infty}^{\infty} |\alpha| |f(x)| e^{-x^2}\, dx$ by proposition 1.1.2.b. By theorem 3.3.3, we have: $\int_{-\infty}^{\infty} |\alpha| |f(x)| e^{-x^2}\, dx = |\alpha| \int_{-\infty}^{\infty} |f(x)| e^{-x^2}\, dx$ $= |a| \|f\|$. Therefore, we have: $\|\alpha f\| = |\alpha| \|f\|.$

iii) $\|f+g\| = \int_{-\infty}^{\infty} |f(x)+g(x)| e^{-x^2}\, dx \le \int_{-\infty}^{\infty} \left(|f(x)|+|g(x)|\right) e^{-x^2}\, dx$ since

$\forall x, \; |f(x)+g(x)| \le |f(x)|+|g(x)|$ by proposition 1.1.2.c and theorem 3.3.4.

$\Rightarrow \int_{-\infty}^{\infty} \left(|f(x)|+|g(x)|\right) e^{-x^2}\, dx = \int_{-\infty}^{\infty} \left(|f(x)| e^{-x^2} + |g(x)| e^{-x^2}\right) dx$

$= \int_{-\infty}^{\infty} |f(x)| e^{-x^2}\, dx + \int_{-\infty}^{\infty} |g(x)| e^{-x^2}\, dx$ by theorem 3.3.3 since f is continuous, $|f|$ is continuous, and e^{-x^2} is continuous.

$\Rightarrow \|f+g\| \le \int_{-\infty}^{\infty} |f(x)| e^{-x^2}\, dx + \int_{-\infty}^{\infty} |g(x)| e^{-x^2}\, dx = \|f\| + \|g\|.$ Q.E.D.