5.8.6 Prove that $\mathbf{R}^n$ is complete in the Euclidean norm. Hint: show that a sequence is Cauchy in $\mathbf{R}^n$ if and only if each of the sequences of components is Cauchy in **R**.

Proof of exercise 5.8.6:

Notation: If $x \in \mathbf{R}^n$ $x = (x_1, ..., x_n)$ where $x_i \in \mathbf{R}$ $\forall i.$

Let $\{x^{(m)}\} \subset \mathbf{R}^n$ be a Cauchy sequence of n-tuples. That is, $\forall \varepsilon > 0$ $\exists N(\varepsilon) \in \mathbf{N}$ $\ni \forall m, m' \geq N$, $\left\| x^{(m)} - x^{(m')} \right\|_2 < \varepsilon.$

Thus, $\forall m, m' \geq N$, $\sqrt{\sum_{k=1}^{n} (x_k^{(m)} - x_k^{(m')})^2} < \varepsilon.$

For each $j \in \{1, ..., n\}$, we have: $\left| x_j^{(m)} - x_j^{(m')} \right| \leq \sqrt{\sum_{k=1}^{n} (x_k^{(m)} - x_k^{(m')})^2} < \varepsilon.$

Therefore, the sequence $\{x_j^{(m)}\}$ is a Cauchy sequence of real numbers. By the axiom of completeness (see section 2.4), $\{x_j^{(m)}\}$ has a limit $x_j \in \mathbf{R}.$

That is, $\forall \varepsilon' > 0$ $\exists N_j(\varepsilon') \in \mathbf{N} \ni \forall m \geq N_j$, $\left\| x_j^{(m)} - x_j \right\|_2 < \dfrac{\varepsilon}{\sqrt{n}}.$

Let $x = (x_1, ..., x_n)$. Since $\forall i$, $x_i \in \mathbf{R}$, $x \in \mathbf{R}^n.$

Then $\forall m \geq \max\{N_1, ..., N_n\}$, we have:

$$\left\| x^{(m)} - x \right\|_2 = \sqrt{\sum_{k=1}^{n} (x_k^{(m)} - x_k)^2} < \sqrt{\sum_{k=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}} \right)^2} = \sqrt{n \frac{\varepsilon^2}{n}} = \sqrt{\varepsilon^2} = \varepsilon.$$

That is, $x^{(m)}$ converges to a point $x \in \mathbf{R}$. Therefore, every Cauchy sequence in $\mathbf{R}^n$ has a limit in $\mathbf{R}^n$.

$\Rightarrow (\mathbf{R}^n, \rho_2)$ is a complete metric space. Q.E.D.