

5.8.4 Let V be a vector space with norm $\|\cdot\|$.

5.8.4.a Prove that for all $v \in V$ and $w \in V$,

$$|\|v\| - \|w\|| \leq \|v - w\|.$$

Proof of exercise 5.8.4.a:

$\|v\| = \|v - w + w\| \leq \|v - w\| + \|w\| \Rightarrow \|v\| - \|w\| \leq \|v - w\|$ by the triangle inequality for norms.

$\|w\| = \|w - v + v\| \leq \|w - v\| + \|v\| \Rightarrow \|w\| - \|v\| \leq \|w - v\| = \|v - w\|$ by the triangle inequality and homogeneity property of norms.

Therefore, $|\|v\| - \|w\|| = \max\{\|v\| - \|w\|, \|w\| - \|v\|\} \leq \|v - w\|$. Q.E.D.

5.8.4.b Suppose that $v_n \rightarrow v$ in V . Prove that $\|v_n\| \rightarrow \|v\|$.

Proof of exercise 5.8.4.b:

Suppose that $v_n \rightarrow v$ in V . Then $\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N, \|v_n - v\| < \varepsilon$.

From 5.8.4.a, we have: $|\|v_n\| - \|v\|| \leq \|v_n - v\|$. Thus, we have that $\forall \varepsilon > 0$

$\exists N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N, |\|v_n\| - \|v\|| \leq \|v_n - v\| < \varepsilon$. Therefore, $\|v_n\| \rightarrow \|v\|$ as $n \rightarrow \infty$.

Q.E.D.