5.8.2 Which of the following subsets of $C(\mathbf{R})$ are vector spaces?

5.8.2.a $C_b(\mathbf{R})$, the bounded continuous functions on **R**.

Answer to exercise 5.8.2.a:

Take two such functions f and g and a scalar $\alpha \in \mathbf{R}$.

Since f is bounded, $\exists M_1 > 0 \ni |f(x)| < M_1 \ \forall x \in \mathbf{R}$.

Since g is bounded, $\exists M_2 > 0 \ni |g(x)| < M_2 \ \forall x \in \mathbf{R}$.

Therefore, $|f(x) + \alpha g(x)| \leq |f(x)| + |a||g(x)| \leq M_1 + \alpha M_2 \equiv M_3$ by proposition 1.1.2.b and 1.1.2.c. Therefore, $f + \alpha g$ is bounded.

By theorem 3.1.1.a and 3.1.1.b, $f + \alpha g$ is continuous on $\mathbf{R}$.

Therefore, $(f + \alpha g) \in C_b(\mathbf{R})$. Thus, $C_b(\mathbf{R})$ is a subspace of $C(\mathbf{R})$ and is thus a vector space.

5.8.2.b $C_0(\mathbf{R})$, the continuous functions that go to zero at $\pm\infty$.

Answer to exercise 5.8.2.b:

Take two such functions f and g and a scalar $\alpha \in \mathbf{R}$.

We know that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$

Therefore, by theorems 2.2.3 and 2.2.4, we have that:

$\lim_{x\to\infty}[f(x) + \alpha g(x)] = \lim_{x\to\infty} f(x) + \alpha \lim_{x\to\infty} g(x) = 0 + \alpha 0 = 0$

Therefore, $(f + \alpha g) \in C_0(\mathbf{R})$.

5.8.2.c The continuous functions that go to 1 at $\pm\infty$.

Answer to exercise 5.8.2.c:

Consider the function $f(x) = 1 + \dfrac{1}{x}$.

$\lim_{x\to\infty} f(x) = 1 = \lim_{x\to-\infty} f(x)$

Take $\alpha = 2 \Rightarrow \lim_{x\to\infty} 2f(x) = 2\lim_{x\to\infty} f(x) = 2 \neq 1$

Therefore, $\alpha f \notin \{$Continuous functions that go to 1 at $\pm\infty\}$

Hence, this is not a vector space.

5.8.2.d $C^{(1)}(\mathbf{R})$, the continuously differentiable functions on **R**.

Answer to exercise 5.8.2.d:

By theorem 4.1.2, if we suppose that f and g are differentiable on **R**, then for any constant α, $f + \alpha g$ is differentiable on **R** and $(f + \alpha g)' = f' + \alpha g'$

Since $f \in C^{(1)}(\mathbf{R})$, we have that f' is continuous on **R.**

Since $g \in C^{(1)}(\mathbf{R})$, we have that $\alpha g'$ is continuous on **R** by theorem 3.1.1.b

Therefore, by theorem 3.1.1.a, we have that $f' + \alpha g'$ is continuous on **R**.

Therefore, $(f + \alpha g) \in C^{(1)}(\mathbf{R})$. i.e. $C^{(1)}(\mathbf{R})$ is a subspace of $C(\mathbf{R})$ and is thus a vector space.

5.8.2.e The continuous functions on **R** that vanish at $x = 5$.

Answer to exercise 5.8.2.e:

Take two such functions f and g and a scalar $\alpha \in \mathbf{R}$.

By definition of addition of functions, we have: $(f + \alpha g)(5) = f(5) + \alpha g(5)$ $= 0 + \alpha 0 = 0$.

Therefore, $(f + \alpha g)$ is in this set. i.e. this set is a subspace of $C(\mathbf{R})$ and is thus a vector space.

5.8.2.f The continuous functions on **R** that satisfy $|f(x)| \le ce^{x^2}$ for some $c \in \mathbf{R}$ which can depend on *f*.

Answer to exercise 5.8.2.f:

Take two such functions f and g and a scalar $\alpha \in \mathbf{R}$.

Then we have that $|f(x)| \le c_f e^{x^2}$ and $|g(x)| \le c_g e^{x^2}$

Then $|f(x) + \alpha g(x)| \le |f(x)| + |\alpha||g(x)| \le c_f e^{x^2} + |\alpha| c_g e^{x^2} = (c_f + |\alpha| c_g) e^{x^2}$

Let $c_{f+\alpha g} = c_f + |\alpha| c_g$. Then, $|f(x) + \alpha g(x)| \le c_{f+\alpha g} e^{x^2}$.

Therefore, $(f + \alpha g)$ is in this set. i.e. this set is a subspace of $C(\mathbf{R})$ and is thus a vector space.