5.8.13 Let A_n be the set of $n \times n$ matrices $A = \{a_{ij}\}$. Define

$$\|A\| \equiv \sup_{ij} |a_{ij}|.$$

5.8.13.a Explain why A_n is a vector space.

Answer to exercise 5.8.13.a:

V is a vector space if the nine properties on page 211 hold.

1. Take $A, A', A'' \in A_n \Rightarrow A + A' = [a_{ij} + a_{ij}'] \in A_n$
2. $A + A' = [a_{ij} + a_{ij}'] = [a_{ij}' + a_{ij}] = A' + A$
3. $(A + A') + A'' = [(a_{ij} + a_{ij}') + a_{ij}''] = [a_{ij} + (a_{ij}' + a_{ij}'')] = A + (A' + A'')$
4. $0 = [0]$
5. $A = [a_{ij}] \Rightarrow \exists -A = [-a_{ij}] \ni A + (-A) = [a_{ij} - a_{ij}] = [0] = 0$
6. Take $\alpha, \beta \in \mathbf{R}$ arbitrary. $\alpha A = [\alpha a_{ij}] \in A_n$
7. $(\alpha + \beta)A = [(\alpha + \beta)a_{ij}] = [\alpha a_{ij} + \beta a_{ij}] = \alpha A + \alpha B$
8. $\alpha(\beta A) = [\alpha(\beta a_{ij})] = [(\alpha\beta)a_{ij}] = (\alpha\beta)A$
9. $\alpha(A + A') = [\alpha(a_{ij} + a_{ij}')] = [\alpha a_{ij} + \alpha a_{ij}'] = \alpha A + \alpha A'$

Therefore, A_n is a vector space.

5.8.13.b Prove that $\|\cdot\|$ is a norm.

Proof of exercise 5.8.13.b:

Let $A = 0 \Rightarrow \quad \|A\| = \sup_{ij} |0| = 0$

Suppose $A \neq 0 \Rightarrow \exists\ i^*, j^* \ni a_{i^*j^*} \neq 0 \Rightarrow |a_{i^*j^*}| > 0$

$\Rightarrow \|A\| = \sup_{ij} |a_{ij}| \geq |a_{i^*j^*}| > 0$

Take $\alpha \in \mathbf{R}. \Rightarrow \|\alpha A\| = \sup_{ij} |\alpha a_{ij}| = |\alpha| \sup_{ij} |a_{ij}| = |\alpha| \|A\|$

Let $A, A' \in A_n$.

$\|A + A'\| = \sup_{ij} |a_{ij} + a_{ij}'|$

$\forall i, j,$ we have that $|a_{ij} + a_{ij}'| \leq |a_{ij}| + |a_{ij}'|$

We also know that $|a_{ij}| \leq \sup_{ij} |a_{ij}|$ and $|a_{ij}'| \leq \sup_{ij} |a_{ij}'|$

$\Rightarrow |a_{ij} + a_{ij}'| \leq \sup_{ij} |a_{ij}| + \sup_{ij} |a_{ij}'|$

Since this holds $\forall i, j,$ we have that

$\|A + A'\| = \sup_{ij} |a_{ij} + a_{ij}'| \leq \sup_{ij} |a_{ij}| + \sup_{ij} |a_{ij}'| = \|A\| + \|A'\|$

Therefore, $\|\cdot\|$ is a norm. Q.E.D.

5.8.13.c Prove that A_n is complete in the norm $\| \cdot \|$.

Proof of exercise 5.8.13.c:

Let $\{A^{(n)}\} \subset A_n$ be a Cauchy sequence of matrices.

Then $\forall \varepsilon > 0 \; \exists N(\varepsilon) \in \mathbf{N} \ni \forall m, n \geq N, \; \left\| A^{(m)} - A^{(n)} \right\| < \varepsilon.$

We also know that $\left\| A^{(m)} - A^{(n)} \right\| = \sup_{ij} \left| a_{ij}{}^{(m)} - a_{ij}{}^{(n)} \right| < \varepsilon.$

$\forall i, j,$ we have: $\left| a_{ij}{}^{(m)} - a_{ij}{}^{(n)} \right| \leq \sup_{ij} \left| a_{ij}{}^{(m)} - a_{ij}{}^{(n)} \right| < \varepsilon.$

Therefore, $\{a_{ij}{}^{(n)}\}$ is a Cauchy sequence of real numbers.

$\Rightarrow a_{ij}{}^{(n)} \rightarrow a_{ij} \in \mathbf{R}$ by the axiom of completeness.

Since this holds $\forall i, j,$ let $A = [a_{ij}]$. Take $n \geq N,$

Then $\left\| A^{(n)} - A \right\| = \sup_{ij} \left| a_{ij}{}^{(n)} - a_{ij} \right| < \varepsilon.$ Therefore, $\{A^{(n)}\}$ is an arbitrary Cauchy sequence of elements of A_n which converges to a matrix $A \in A_n$.

$\Rightarrow A_n$ is complete in the norm $\| \cdot \|$. Q.E.D.

5.8.13.d Let B be a fixed element of A_n and define $T(A) = BA$ where BA denotes matrix multiplication. Show that T is a linear transformation on A_n.

Proof of exercise 5.8.13.d:

Consider $A, A' \in A_n$, $\alpha \in \mathbf{R}$.

$T(A + \alpha A') = B(A + \alpha A') = BA + \alpha BA' = T(A) + \alpha T(A').$

Therefore, T is a linear transformation on A_n. Q.E.D.