5.8.12 Let T be a linear transformation from a Banach space to itself and suppose that T is a contraction. What fixed points can T have?

Answer to exercise 5.8.12:

Since T is a contraction on a Banach space (which is complete), it has a unique fixed point by the contraction mapping theorem.

Since T is a linear transformation, $T(0) = T(x - x) = T(x) - T(x) = 0$

I know that the contraction mapping theorem already prescribes that the fixed point is unique, but a two-line proof seems awfully short, so I will prove uniqueness again:

Suppose $\exists\, \bar{x} \in V \ni \bar{x} = T\bar{x}$ but $\bar{x} \neq 0$.

Then $\rho(\bar{x},0) = \rho(T\bar{x},T(0)) \leq \beta\rho(\bar{x},0)$ $\beta < 1$. Since $\bar{x} \neq 0$, $\rho(\bar{x},0) > 0$

$\Rightarrow \dfrac{\rho(\bar{x},0)}{\rho(\bar{x},0)} \leq \beta \dfrac{\rho(\bar{x},0)}{\rho(\bar{x},0)} \Rightarrow 1 \leq \beta$, which is a contradiction. Therefore, there are no fixed points $\bar{x} \neq 0$ of T.

$\Rightarrow$ 0 is the only fixed point of T.