5.8.11 Let c_0 denote the set of sequences, $\{a_j\}$, of real numbers such that $a_j \to 0$ as $j \to \infty$. Define

$$\|\{a_j\}\|_\infty \equiv \sup_j |a_j|.$$

5.8.11.a Explain why c_0 is a normed linear space with the norm $\|\cdot\|_\infty$.

Answer to exercise 5.8.11.a:

Take an element $\{a_j\}$ of c_0. $a_j \to 0$ as $j \to \infty$. In particular, $\{a_j\}$ is a convergent sequence. Therefore, by proposition 2.2.1, $\{a_j\}$ is a bounded sequence. Therefore, $\{a_j\} \in l_\infty$. $\Rightarrow c_0 \subset l_\infty$.

Since l_∞ is a normed linear space with the norm $\|\cdot\|_\infty$, it suffices to show that c_0 is a subspace of l_∞.

Consider two elements $\{a_j\},\{b_j\}$ of c_0 and a scalar $\alpha \in \mathbf{R}$.

Then $\lim_{j\to\infty}(a_j + \alpha b_j) = \lim_{j\to\infty} a_j + \alpha \lim_{j\to\infty} b_j = 0 + \alpha 0 = 0$ by theorems 2.2.3 and 2.2.4.

Therefore, $\{a_j + \alpha b_j\} \in c_0$ and c_0 is a subspace. Therefore, c_0 is a normed linear space with the same norm as the space of which it is a subspace - $\|\cdot\|_\infty$.

5.8.11.b Prove that c_0 is complete. Hint: since $c_0 \subseteq \ell_\infty$, we know that any Cauchy sequence has a limit in ℓ_∞.

Denote $\bar{x} \equiv \{x_j\}$. Consider a Cauchy sequence of elements $\{\bar{a}^{(n)}\}$ of c_0.

That is, $\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbf{N} \ni \forall m,n \geq N,\ \|\bar{a}^{(m)} - \bar{a}^{(n)}\| < \varepsilon.$

But we also know that $\|\bar{a}^{(m)} - \bar{a}^{(n)}\| = \sup_j |a_j^{(m)} - a_j^{(n)}| < \varepsilon.$

In addition, $\forall j,\ |a_j^{(m)} - a_j^{(n)}| \leq \sup_h |a_j^{(m)} - a_j^{(n)}| < \varepsilon$. That is, for each j, the sequence $\{a_j^{(n)}\}$ is a Cauchy sequence of real numbers. Therefore, by the axiom of completeness, $\exists\ a_j \in \mathbf{R} \ni a_j^{(n)} \to a_j$ as $n \to \infty$.

Since this holds $\forall j$, we can let $\bar{a} = \{a_j\}$

Then, we have that, given any $\varepsilon > 0, \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N$, $\|\bar{a}^{(n)} - \bar{a}\| = \sup_j |a_j^{(n)} - a_j| < \varepsilon$. Therefore, $\bar{a}^{(n)} \to \bar{a}$. It remains to show that $\bar{a} \in c_0$.

We know that for each n, $a_j^{(n)} \to 0$ as $j \to \infty$. That is, $\forall \varepsilon > 0\ \exists N_n(\varepsilon) \in \mathbf{N}$ $\ni \forall j \geq N_n,\ |a_j^{(n)}| < \dfrac{\varepsilon}{2}.$

In addition, from earlier, we have that $a_j^{(n)} \to a_j$ for each j. That is,

$\forall \varepsilon > 0\ \exists N_j \in \mathbf{N} \ni \forall n \geq N_j,\ \left|a_j^{(n)} - a_j\right| < \frac{\varepsilon}{2}.$

Take $\varepsilon > 0$ arbitrary. $\forall n \geq N_j$ and $j \geq N_n$, we have:

$\left|a_j\right| < \left|a_j^{(n)} - a_j\right| + \left|a_j^{(n)}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$ That is, $a_j \to$ as $j \to \infty.$ Therefore,

$\bar{a} \in c_0$ and c_0 is therefore complete. Q.E.D.

5.8.11.c Show that the set of sequences which are zero after finitely many terms is dense in c_0.

Answer to exercise 5.8.11.c:

Take an arbitrary element $\bar{a} \in c_0$. $\bar{a} = \{a_1, a_2, a_3, ...\}$

Consider the sequence of sequences $\left\{\bar{h}^{(n)}\right\}$ where:

$\bar{h}^{(1)} = \{a_1, 0, 0, ...\}$; $\bar{h}^{(2)} = \{a_1, a_2, 0, ...\}$; … ; $\bar{h}^{(n)} = \{a_1, a_2, ..., a_n, 0, 0, ...\}$

Therefore, we have: $\bar{a} - \bar{h}^{(n)} = \{0, 0, ..., 0, a_{n+1}, a_{n+2}, ...\}$

Since $a_j \to 0$ as $j \to \infty,$ we have: $\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbf{N} \ni \forall j \geq N,\ \left|a_j\right| < \varepsilon.$

Take $\varepsilon > 0$ arbitrary. Then, $\forall n \geq N,$ the following holds:

$\left\|\bar{a} - \bar{h}^{(n)}\right\| = \sup_{j \geq N}\left|a_j\right| < \varepsilon.$ Therefore, the set of sequences which are zero after finitely many terms is dense in c_0. Q.E.D.

5.8.11.d Show that c_0 is not dense in ℓ_∞.

Answer to exercise 5.8.11.d:

Consider the sequence $\{a_n\} = \{1\}$ $a_n \in l_\infty$, since $\left|a_n\right| \leq 1\ \forall n$. Take any sequence $c = \{c_j\} \in c_0$. $\left\|\{a_n\} - c\right\| = \sup_j\left|1 - c_j\right|$. Since $c_j \to 0$ as $j \to \infty,$ we have that

$\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbf{N} \ni \forall j \geq N,\ \left|c_j\right| < \varepsilon.$ For all such j, we have:

$\left|1 - c_j\right| \geq 1 - \left|c_j\right| > 1 - \varepsilon.$ Therefore, $\left\|\{a_n\} - c\right\| = \sup_j\left|1 - c_j\right| > 1 - \varepsilon$

i.e. c does not converge to $\{a_n\}$. i.e. Not every element of l_∞ can be constructed by a sequence of elements of c_0.