5.8.1 Which of the following subsets of $C[a,b]$ are vector spaces?

5.8.1.a The continuous functions, *f*, which satisfy $f(a)=1$.

Answer to exercise 5.8.1.a:

Take two such functions f and f'. $f(a)=1$ and $f'(a)=1$. But $(f+f')(a)=f(a)+f'(a)=1+1=2\neq 1$. Therefore, this is not a vector space.

5.8.1.b $C^{(1)}[a,b]$.

Answer to exercise 5.8.1.b:

By theorem 4.1.2, if we suppose that f and g are differentiable on $[a,b]$, then for any constant α, $f+\alpha g$ is differentiable on $[a,b]$ and $(f+\alpha g)'=f'+\alpha g'$

Since $f\in C^{(1)}[a,b]$, we have that f' is continuous on $[a,b]$.

Since $g\in C^{(1)}[a,b]$, we have that $\alpha g'$ is continuous on $[a,b]$ by theorem 3.1.1.b

Therefore, by theorem 3.1.1.a, we have that $f'+\alpha g'$ is continuous. Therefore, $(f+\alpha g)\in C^{(1)}[a,b]$. i.e. $C^{(1)}[a,b]$ is a subspace of $C[a,b]$ and therefore is a vector space.

5.8.1.c The continuous functions, *f*, which satisfy $\int_a^b f(x)dx=0$.

Answer to exercise 5.8.1.c:

Consider two such functions f and g and a scalar $\alpha\in\mathbf{R}$.

$$\int_a^b \left[f(x)+\alpha g(x)\right]dx=\int_a^b f(x)dx+\alpha\int_a^b g(x)dx=0+\alpha(0)=0$$

where theorem 3.3.3 was used in the first step.

Therefore, this is a subspace of $C[a,b]$, i.e. it is a vector space.

5.8.1.d The functions $f\in C^{(2)}[a,b]$ that satisfy

$$f''(x)+(2x^2+1)f'(x)+(\sin x)f(x)=0$$

Answer to exercise 5.8.1.d:

Consider two such functions f and g and a scalar $\alpha\in\mathbf{R}$.

$$\begin{aligned}
&(f+\alpha g)''(x)+(2x^2+1)(f+\alpha g)'(x)+(\sin x)(f+\alpha g)(x)\\
&=f''(x)+\alpha g''(x)+(2x^2+1)f'(x)+(2x^2+1)\alpha g'(x)+(\sin x)f(x)+(\sin x)\alpha g(x)\\
&=f''(x)+(2x^2+1)f'(x)+(\sin x)f(x)+\alpha[g''(x)+(2x^2+1)g'(x)+(\sin x)g(x)]\\
&=0+\alpha 0=0
\end{aligned}$$

where theorem 4.1.2 was used extensively.

Therefore, this is a subspace of $C[a,b]$, i.e. it is a vector space.