Michael Powell; 603-210-926
Math 131B; Prof. Eskin
University of California at Los Angeles
February 6th, 2005
Problem Set 5

5.7.11 Let (M_i, ρ_i), $i = 1, ..., N,$ be a finite collection of complete metric spaces. Let M be the product of the spaces M_i; that is, M consists of the N-tuples $(x_1, x_2, ..., x_N)$ with $x_i \in M_i$ for each i. For two such N-tuples, $x = (x_1, x_2, ..., x_N)$ and $y = (y_1, y_2, ..., y_N)$, define

$$\rho(x, y) = \sum_{i=1}^{N} \rho_i(x_i, y_i)$$

Prove that (M, ρ) is a complete metric space.

Proof of exercise 5.7.11

Let $\{x^{(n)}\} \subset M$ be a Cauchy sequence. That is, $\forall \varepsilon > 0\ \exists\ N(\varepsilon) \in \mathbf{N} \ni \forall m, n \geq N,$ we have $\rho(x^{(m)}, x^{(n)}) < \varepsilon.$ Thus, $\forall m, n \geq N\ \sum_{i=1}^{N} \rho_i(x_i^{(m)}, x_i^{(n)}) < \varepsilon.$

For each $j \in \{1, ..., N\},$ we have $\rho_j(x_j^{(m)}, x_j^{(n)}) < \sum_{i=1}^{N} \rho_i(x_i^{(m)}, x_i^{(n)}) < \varepsilon.$

That is, for each $j,$ the sequence $\{x_j^{(n)}\}$ is a Cauchy sequence. Since $\{x_j^{(n)}\}$ is a Cauchy sequence of $M_j,$ a complete metric space, it converges to a point $x_j \in M_j$. That is, $\forall \varepsilon' > 0\ \exists N_j(\varepsilon') \in \mathbf{N} \ni \forall n \geq N_j,\ \rho_j(x_j^{(n)}, x_j) < \frac{\varepsilon'}{N}.$

Let $x = (x_1, ..., x_N).$ Then $\forall n \geq \max\{N_1, ..., N_N\}$ we have:

$\rho(x^{(n)}, x) = \sum_{i=1}^{N} \rho_i(x_i^{(n)}, x_i) < \sum_{i=1}^{N} \frac{\varepsilon'}{N} = \varepsilon'.$ That is, $x^{(n)} \to x \in M$ as $n \to \infty.$

Therefore, (M, ρ) is a complete metric space. Q.E.D.