5.7.5 Which of the following subsets of $C[a,b]$ are complete metric spaces with the metric ρ_∞?

A metric space (M,ρ) is complete if every Cauchy sequence $\{x_n\} \subset M$ converges to an element $x \in M$. Since we know that $C[a,b]$ is a complete metric space, with respect to the ρ_∞ metric (by Theorem 5.3.3), we know that every Cauchy sequence $\{f_n\} \subset C[a,b]$ converges to a function $f \in C[a,b]$.

Consider a subset $F \subset C[a,b]$. F is complete (with respect to ρ_∞ if every Cauchy sequence $\{f_n\} \subset F$ converges to a function $f \in F$. Since $F \subset C[a,b]$., we know that every Cauchy sequence of points $\{f_n\} \subset F$ converges to a function $f \in C[a,b]$. If we can show that $f \in F$., then it follows that F is a complete metric space. If F is a closed set, then it contains all its limit points. (i.e. $f \in F$.). Thus, it suffices to determine whether or not each set is closed.

In this problem, I will be rather loose with the definitions of uniform convergence and convergence in the ρ_∞ metric, since by proposition 5.3.2, the two are equivalent.

5.7.5.a $F = \{f \in C[a,b] \mid f(x) > 0 \text{ for } x \in [a,b]\}$

Consider the sequence $f_n(x) = \frac{1}{n}\ \forall x \in [a,b]$. $f_n(x) = \frac{1}{n} > 0\ \forall x \in [a,b]\ \forall n.$

Therefore, $f_n \in F\ \forall n$. This sequence clearly converges uniformly to the zero function $[a,b]$. Choose $N = \frac{1}{\varepsilon}$ to see this.

By proposition 5.3.2, since $f_n \to 0$ uniformly on $[a,b]$, we have that $\rho_\infty(f_n, 0) \to 0$ as $n \to \infty$. (And hence $\{f_n\}$ is a Cauchy sequence) But $0 \notin F$.

Therefore, we have a Cauchy sequence of functions in F which converges to a function not in F. i.e. F is not a closed set

$\Rightarrow (F, \rho_\infty)$ is not a complete metric space.

5.7.5.b $F = \{f \in C[a,b] \mid f(a) = 0\}$

Consider a sequence of functions $\{f_n\} \subset F$. It must be the case that $\forall n,\ f_n(a) = 0$. Suppose $f_n \to f \notin F$ uniformly on $[a,b]$ as $n \to \infty$. Then $f(a) \neq 0$.

But $|f_n(a) - f(a)| = |0 - f(a)| = |f(a)| > 0\ \forall n$. Thus, $\sup_{x \in [a,b]} |f_n(x) - f(x)|$ $\geq |f(a)| > 0\ \forall n$, which contradicts $f_n \to f$ uniformly on $[a,b]$. Therefore, it must be the case that, if $f_n \to f$ uniformly on $[a,b]$ (i.e. $\{f_n\}$ is Cauchy with respect to ρ_∞), then $f(a) = 0 \Rightarrow f \in F$. Therefore, F is a closed set.

$\Rightarrow (F, \rho_\infty)$ is a complete metric space.

5.7.5.c $F = \{f \in C[a,b] \mid f(x) = 0 \text{ for } a < c \le x \le d < b\}$

Consider a sequence of functions $\{f_n\} \subset F$. It must be the case that for all $e \in [c,d], \forall n, f_n(e) = 0$. Suppose $f_n \to f \notin F$ uniformly on $[a,b]$ as $n \to \infty$. Then $\exists\, e \in [c,d] \ni f(e) \neq 0$.

But $|f_n(e) - f(e)| = |0 - f(e)| = |f(e)| > 0\ \forall n$. Thus, $\sup_{x\in[a,b]} |f_n(x) - f(x)|$ $\ge |f(e)| > 0\ \ \forall n,$ which contradicts $f_n \to f$ uniformly on $[a,b]$. Therefore, it must be the case that, if $f_n \to f$ uniformly on $[a,b]$ (i.e. $\{f_n\}$ is Cauchy with respect to ρ_∞), then $f(e) = 0\ \forall e \in [c,d] \Rightarrow f \in F$. Therefore, F is a closed set.

$\Rightarrow (F, \rho_\infty)$ is a complete metric space.

5.7.5.d $F = \{f \in C[a,b] \mid |f(x)| \le 2 + f(x)^2 \text{ for } x \in [a,b]\}$

Let $\{f_n\} \subset F$ be a sequence of functions. Suppose $f_n \to f$. ($\{f_n\}$ is thus a Cauchy sequence) We want to show that $f \in F$.

Let n be arbitrary. Take $f_n \in F$. $|f_n(x)| \le 2 + f_n(x)^2 \Rightarrow f_n(x)^2 - |f_n(x)| \ge -2$

Define $g_n(x) = f_n(x)^2 - |f_n(x)|$. Since f_n is continuous on $[a,b]$, $|f_n|$ is continuous on $[a,b]$, and thus $f_n(x)^2 - |f_n(x)|$ is continuous on $[a,b]$ by theorem 3.1.1. Therefore $g_n(x)$ is continuous on $[a,b]$.

$\Rightarrow g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}\left(f_n(x)^2 - |f_n(x)|\right) = f(x)^2 - |f(x)|.$

If we can show that $g(x) \ge -2$, we are done.

Take $x \in [a,b]$ arbitrary. We have $g_n(x) \ge -2$. Since $g_n(x) \to g(x)$, by exercise 2.1.6, we have that $g(x) \ge -2$. But since x was arbitrary, we have that $\forall x \in [a,b]$, $g(x) \ge -2$. Therefore $f \in F \Rightarrow F$ is a closed set. Thus, every Cauchy sequence of functions in F converges to a function in F.

$\Rightarrow (F, \rho_\infty)$ is a complete metric space.