5.7.4 Which of the following subsets of $\mathbf{R}^2$ are complete metric spaces with the Euclidean metric?

A metric space (M,ρ) is complete if every Cauchy sequence $\{x_n\} \subset M$ converges to a point $x \in M$. Since we know that $\mathbf{R}^2$ is a complete metric space (by the axiom of completeness), we know that every Cauchy sequence $\{(x_m, y_m)\} \subset \mathbf{R}^2$ converges to a point $(x,y) \in \mathbf{R}^2$.

Consider a subset $A \subset \mathbf{R}^2$. A is complete (with respect to ρ) if every Cauchy sequence $\{(x_m, y_m)\} \subset A$ converges to a point $(x,y) \in A$. Since $A \subset \mathbf{R}^2$., we know that every Cauchy sequence of points $\{(x_m, y_m)\} \subset A$ converges to a point $(x,y) \in \mathbf{R}^2$. If we can show that $(x,y) \in A$., then it follows that *A* is a complete metric space. If *A* is a closed set, then it contains all its limit points. (i.e. $(x,y) \in A$.). Thus, it suffices to determine whether or not each set is closed.

5.7.4.a $A = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$

Consider the sequence $a_n = \left(0, 1 - \frac{1}{n}\right)$. $\forall n$, we have: $0^2 + \left(1 - \frac{1}{n}\right)^2 = 1 - \frac{2}{n} + \frac{1}{n^2}$ $\leq 1 - \frac{2}{n} + \frac{1}{n} = 1 - \frac{1}{n} \leq 1$. i.e. $a_n \in A$. As proven in the first lecture, a sequence of points in $\mathbf{R}^2$ converges if and only if each element converges. Thus, since $0 \to 0$ as $n \to \infty$ and $1 - \frac{1}{n} \to 1$ as $n \to \infty$, we have that $a_n \to (0,1)$. But
$0^2 + 1^2 = 1 \geq 1 \Rightarrow (0,1) \notin A$.
Therefore, we have a Cauchy sequence of points in *A* which converges to a point not in *A*. i.e. *A* is not a closed set.
Therefore, (A, ρ_2) is not a complete metric space.

5.7.4.b $A = \{(x,y) \in \mathbf{R}^2 \mid x \geq 1 \text{ and } y \leq -2\}$

Consider the sets $B = \{(x,y) \in \mathbf{R}^2 \mid x \geq 1\}$ and $C = \{(x,y) \in \mathbf{R}^2 \mid y \leq -2\}$.
$A = B \bigcap C$. I will prove that both B and C are closed sets. The finite intersection of two closed sets is closed (see any introductory toplogy textbook).

Consider the set $\mathbf{R}^2 \setminus B = \{(x,y) \in \mathbf{R}^2 \mid x < 1\}$. Take any point $d \in \mathbf{R}^2 \setminus B$.
$d = (x,y)$ where $x < 1$. The open ball $B_{1-\frac{x}{2}}(d) \subset \mathbf{R}^2 \setminus B$. That is, for every $d \in \mathbf{R}^2 \setminus B$, we can construct an open ball around it which is completely contained within the set. i.e. $\mathbf{R}^2 \setminus B$ is an open set. Therefore, *B* is a closed set.

Consider the set $\mathbf{R}^2 \setminus C = \{(x,y) \in \mathbf{R}^2 \mid y > -2\}$. Take any point $d \in \mathbf{R}^2 \setminus C$. $d = (x,y)$ where $y > -2$. The open ball $B_{\frac{y}{2}+1}(d) \subset \mathbf{R}^2 \setminus C$. That is, for every $d \in \mathbf{R}^2 \setminus C$, we can construct an open ball around it which is completely contained within the set. i.e. $\mathbf{R}^2 \setminus C$ is an open set. Therefore, C is a closed set. Since B and C are closed sets, we have that $B \bigcap C$ is a closed set. Therefore, A is a closed set.

$\Rightarrow (A, \rho_2)$ is a complete metric space.

5.7.4.c $A = \{(x,y) \in \mathbf{R}^2 \mid y \in \mathbf{N}\}$

Let $\{p_n\} \subset A$. Suppose $p_n \to p$ as $n \to \infty$. ($\{p_n\}$ is thus Cauchy). As proven in class, if $p_n = (x_n, y_n) \to (x,y) = p$, it must be the case that $x_n \to x$ and $y_n \to y$. Since $y_n \to y$, $\forall \varepsilon > 0 \ \exists\ N(\varepsilon) \in \mathbf{N} \ni \forall\ n \geq N$, $|y_n - y| < \varepsilon$. Suppose we have that $\varepsilon < 1$. Then it must be the case that $\forall n \geq N(\varepsilon)$, $y_n = k \in \mathbf{N}$ $\Rightarrow y_n \to k$ as $n \to \infty$.

Therefore, we have that $p_n = (x_n, y_n) \to (x,k) = p$ for some $k \in \mathbf{N}$. Thus, $p \in A$. i.e. A is a closed set.
Therefore, (A, ρ_2) is a complete metric space.

5.7.4.d $A = \{(x,y) \in \mathbf{R}^2 \mid f(x,y) = 0\}$ where f is continuous on $\mathbf{R}^2$.

Lemma 5.7.4.d.a:

If $a_n = 0\ \forall n$ and $a_n \to b$, then $b = 0$.

Proof of lemma 5.7.4.d.a:

In order to get a contradiction, suppose $b \neq 0$. Then $|b| > 0$.

Since $a_n \to b$, we have that $\forall \varepsilon > 0 \ \exists\ N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N$, $|a_n - b| < \varepsilon$.

Choose $\varepsilon = \frac{|b|}{2}$. Then $\forall n \geq N\left(\frac{|b|}{2}\right)$, we have $|b| - |a_n| \leq |a_n - b| < \frac{|b|}{2}$

$\Rightarrow |b| - \frac{|b|}{2} = \frac{|b|}{2} < |a_n|$. i.e. $a_n \neq 0$ for some n, which is a contradiction. $\rightarrow\leftarrow$

Therefore, $b = 0$. Q.E.D.

Answer to exercise 5.7.4.d:

Take $\{p_n\} \subset A \ni p_n \to p$ as $n \to \infty$.

Since $p_n \to p$ and since f is continuous on $\mathbf{R}^2$, it follows that $f(p_n) \to f(p)$.

Since $\forall n$, $p_n \in A$, $f(p_n) = 0$. Thus, by lemma 5.7.4.d.a, we have that $f(p) = 0$.

In other words, $p \in A$. Therefore, A is a closed set.

$\Rightarrow (A, \rho_2)$ is a complete metric space.