5.7.3 Which of the following subsets of **R** are complete metric spaces with the Euclidean metric?

A metric space (M,ρ) is complete if every Cauchy sequence $\{x_n\} \subset M$ converges to a point $x \in M$. Since we know that $\mathbf{R}$ is a complete metric space (by the axiom of completeness, section 2.4), we know that every Cauchy sequence $\{x_m\} \subset \mathbf{R}$ converges to a point $x \in \mathbf{R}$.

Consider a subset $A \subset \mathbf{R}$. A is complete (with respect to ρ) if every Cauchy sequence $\{x_n\} \subset A$ converges to a point $x \in A$. Since $A \subset \mathbf{R}$, we know that every Cauchy sequence of points $\{x_n\} \subset A$ converges to a point $x \in \mathbf{R}$. If we can show that $x \in A$, then it follows that *A* is a complete metric space. If *A* is a closed set, then it contains all its limit points. (i.e. $x \in A$). Thus, it suffices to determine whether or not each set is closed.

5.7.3.a $[-1,6]$

Clearly, $[-1,6] \subset \mathbf{R}$ is a closed subset. To show this more rigorously, consider the complement: $(-\infty,-1) \cup (6,\infty)$. Take $x \in (-\infty,-1) \cup (6,\infty)$.

Case I: Suppose $x \in (-\infty,-1)$. Then $B_{\frac{-1-x}{2}}(x) \subset (-\infty,-1)$. That is, for every $x \in (-\infty,-1)$, we can construct an open ball around it which is contained within the set. i.e. $(-\infty,-1)$ is an open set.

Case II: Suppose $x \in (6,\infty)$. Then $B_{\frac{x-6}{2}}(x) \subset (6,\infty)$. That is, for every $x \in (6,\infty)$, we can construct an open ball around it which is contained within the set. i.e. $(6,\infty)$ is an open set.

Since $(-\infty,-1) \cup (6,\infty)$ is the finite union of open sets, it is an open set (see any elementary topology text). Thus, $[-1,6]$ is the complement of an open set and is thus closed.

Therefore, $([-1,6],\rho_2)$ is a complete metric space.

5.7.3.b $[0,\infty)$

Consider the complement: $(-\infty,0)$. Suppose $x \in (-\infty,0)$. Then $B_{\frac{x}{2}}(x) \subset (-\infty,0)$.

That is, for every $x \in (-\infty,0)$, we can construct an open ball around it which is contained within the set. i.e. $(-\infty,0)$ is an open set. Therefore, $[0,\infty)$ is the complement of an open set and is thus closed.

Therefore, $([0,\infty),\rho_2)$ is a complete metric space.

5.7.3.c $(0,\infty)$

Consider the sequence of points $x_n = \frac{1}{n}$. $\forall n$, we have $x_n \in (0,\infty)$. $\{x_n\}$ is a Cauchy sequence (since it is a convergent sequence). We know that $x_n \to 0 \notin (0,\infty)$ as $n \to \infty$. Therefore, we have a Cauchy sequence of points in $(0,\infty)$ which converges to a point not in $(0,\infty)$. i.e. $(0,\infty)$ is not a closed set.

Therefore, $((0,\infty),\rho_2)$ is not a complete metric space.

5.7.3.d **Q**

Consider the sequence of points $\{x_n\} = \{3, 3.1, 3.14, 3.141,...\}$. This sequence is a Cauchy sequence (see page 47). Therefore, it converges. But we see that $x_n \to \pi \notin \mathbf{Q}$ as $n \to \infty$. Therefore, we have a Cauchy sequence of points in **Q** which converges to a point not in **Q**. i.e. **Q** is not a closed set.

Therefore, $(\mathbf{Q},\rho_2)$ is not a complete metric space.

5.7.3.e **N**

Since $\forall x, y \in \mathbf{N} \ni x \neq y$, we have that $\rho(x,y) \geq 1$, if we pick $\varepsilon < 1$, it follows that if $x, y \in \mathbf{N}$, $\rho(x,y) < \varepsilon \Rightarrow x = y$. That is, in order for a sequence of natural numbers to be a Cauchy sequence for some N, it must be the case that $\forall n \geq N$, $x_n = a \in \mathbf{N}$. This sequence clearly converges to $a \in \mathbf{N}$. Therefore, **N** is a closed set with respect to the ρ_2 metric.

Therefore, $(\mathbf{N},\rho_2)$ is a complete metric space.