5.7.2 Let (M, ρ) be a metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (M, ρ). Prove that $\lim_{n\to\infty} \rho(x_n, y_n)$ exists.

Proof of exercise 5.7.2:

Since ρ is a metric, we have (by the triangle inequality):

$\rho(x_n, y_n) \le \rho(x_n, x_m) + \rho(x_m, y_m) + \rho(y_m, y_n)$

$\Rightarrow \rho(x_n, y_n) - \rho(x_m, y_m) \le \rho(x_m, x_n) + \rho(y_m, y_n)$ and

$\rho(x_m, y_m) \le \rho(x_m, x_n) + \rho(x_n, y_n) + \rho(y_n, y_m)$

$\Rightarrow \rho(x_m, y_m) - \rho(x_n, y_n) \le \rho(x_m, x_n) + \rho(y_m, y_n)$

Thus, $|\rho(x_m, y_m) - \rho(x_n, y_n)| \le \rho(x_m, x_n) + \rho(y_m, y_n)$

Since ρ is a real valued function, $\rho(x_m, y_m), \rho(x_n, y_n) \in \mathbf{R}$.

Denote $a_m \equiv \rho(x_m, y_m)$ and $a_n \equiv \rho(x_n, y_n)$.

Since $\{x_n\}$ is a Cauchy sequence, $\forall \varepsilon > 0\ \exists\ N_1(\varepsilon) \in \mathbf{N} \ni \forall\ n, m \ge N_1$, we have:

$\rho(x_m, x_n) < \frac{\varepsilon}{2}$.

Since $\{y_n\}$ is a Cauchy sequence, $\forall \varepsilon > 0\ \exists\ N_2(\varepsilon) \in \mathbf{N} \ni \forall\ n, m \ge N_2$, we have:

$\rho(y_m, y_n) < \frac{\varepsilon}{2}$.

Take $\varepsilon > 0$ arbitrary. Let $N = \max\{N_1, N_2\}$. Then $\forall m, n \ge N$,

$|a_m - a_n| \le \rho(x_m, x_n) + \rho(y_m, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. That is, $\{a_n\}$ is a Cauchy sequence of real numbers. By the axiom of completeness, $\{a_n\}$ converges to a finite limit

i.e. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \rho(x_n, y_n)$ exists. Q.E.D.