5.7.1 Show that a convergent sequence in a metric space is a Cauchy sequence.

Proof of exercise 5.7.1:

Consider the metric space (M, ρ).

Suppose the sequence $\{x_n\} \subset M$ converges to $x \in M$.

Then, $\forall \varepsilon > 0 \ \exists \ N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N, \rho(x_n, x) < \frac{\varepsilon}{2}$.

$\rho(x_m, x_n) \leq \rho(x_m, x) + \rho(x_n, x)$ by the triangle inequality.

Take $m, n \geq N$.

$\Rightarrow \rho(x_m, x_n) \leq \rho(x_m, x) + \rho(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

That is, $\forall \varepsilon > 0 \ \exists \ N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N, \ \rho(x_m, x_n) < \varepsilon$. i.e. $\{x_n\}$ is a Cauchy sequence. Q.E.D.