5.6.9 Let (M, ρ) be a metric space. A point $d \in M$ is called a **limit point** of a sequence $\{x_n\}$ if for every $\varepsilon > 0$ there is an n so that $\rho(d, x_n) \leq \varepsilon$. Prove that d is a limit point of $\{x_n\}$ if and only if $\{x_n\}$ has a subsequence which converges to d.

Proof of exercise 5.6.9 ($\Rightarrow$):

Let $\{x_n\} \subset M$ be a sequence of points. Suppose d is a limit point of $\{x_n\}$.

Then, $\forall \varepsilon > 0 \; \exists \; n \in \mathbf{N} \ni \rho(d, x_n) \leq \varepsilon$.

Construct a sequence $\{x_{n_k}\}$ as follows:

Let $\varepsilon = 1$. Then $\exists \; n_1 \in N \ni \rho(d, x_{n_1}) \leq 1$.

Let $\varepsilon = \dfrac{1}{2}$. Then $\exists \; n_2 \in N \ni \rho(d, x_{n_2}) \leq \dfrac{1}{2}$.

$\vdots$

Let $\varepsilon = \dfrac{1}{k}$. Then $\exists \; n_k \in N \ni \rho(d, x_{n_k}) \leq \dfrac{1}{k}$.

$\vdots$

Consider this sequence $\{x_{n_k}\}$. $\forall \varepsilon > 0$, let $K \geq \dfrac{1}{\varepsilon}$. Then, $\forall k \geq K$, we have:

$\rho(d, x_{n_k}) < \varepsilon$. i.e. $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to d. Q.E.D.

Proof of exercise 5.6.9 $(\Leftarrow)$:

Take the sequence $\{x_n\} \subset M$. Suppose $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to d. That is, $\forall \varepsilon > 0 \; \exists \; K \in \mathbf{N} \ni \forall k \geq K, \rho(d, x_{n_k}) < \varepsilon$.

In particular, $\forall \varepsilon > 0 \; \exists \; n_k \in \mathbf{N} \ni \rho(d, x_{n_k}) < \varepsilon$ where $x_{n_k} \in \{x_n\}$. i.e. d is a limit point of the sequence $\{x_n\}$. Q.E.D.