5.6.8 Suppose that $x_n \to x$ and $y_n \to y$ in metric space (M, ρ). Prove that $\lim_{n\to\infty} \rho(x_n, y_n) = \rho(x, y)$.

Lemma 5.6.8.a:

$$\left|\rho(x_n, y_n) - \rho(x, y)\right| \le \rho(x_n, x) + \rho(y_n, y)$$

Proof of lemma 5.6.8.a:

$\rho(x_n, y_n) \le \rho(x_n, x) + \rho(x, y) + \rho(y_n, y)$ by properties (b) and (c) of metrics.

$\Rightarrow \rho(x_{n,} y_n) - \rho(x, y) \le \rho(x_n, x) + \rho(y_n, y)$

$\rho(x, y) \le \rho(x_n, x) + \rho(x_n, y_n) + \rho(y_n, y)$ by properties (b) and (c) of metrics.

$\Rightarrow \rho(x, y) - \rho(x_n, y_n) \le \rho(x_n, x) + \rho(y_n, y)$

Thus, $\left|\rho(x_n, y_n) - \rho(x, y)\right| = \begin{cases} \rho(x_n, y_n) - \rho(x, y) \ \text{if} \ \ \rho(x_n, y_n) - \rho(x, y) \ge 0 \\ \rho(x, y) - \rho(x_n, y_n) \ \text{if} \ \ \rho(x_n, y_n) - \rho(x, y) < 0 \end{cases}$, both of which are $\le \rho(x_n, x) + \rho(y_n, y)$. Q.E.D.

Proof of exercise 5.6.8:

Since $x_n \to x$, $\forall \varepsilon > 0 \ \exists \ N_1(\varepsilon) \in \mathbf{N} \ni \forall \ n \ge N_1, \rho(x_n, x) < \frac{\varepsilon}{2}$.

Since $y_n \to y$, $\forall \varepsilon > 0 \ \exists \ N_2(\varepsilon) \in \mathbf{N} \ni \forall \ n \ge N_2, \rho(y_n, y) < \frac{\varepsilon}{2}$.

Choose $\varepsilon > 0$ arbitrary. Then, $\forall n \ge \max\{N_1, N_2\}$, we have:

$\left|\rho(x_n, y_n) - \rho(x, y)\right| \le \rho(x_n, x) + \rho(y_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ where lemma 5.6.8.a was used in the first step. Thus, $\lim_{n\to\infty} \rho(x_n, y_n) = \rho(x, y)$. Q.E.D.