5.6.6 Prove that $\rho(x,y) \equiv \left|\frac{1}{x}-\frac{1}{y}\right|$ is a metric on $(0,\infty)$.

A metric satisfies the following three properties:
(a) $\forall x,y \in M$, $\rho(x,y) \geq 0$ and $\rho(x,y)=0$ if and only if $x=y$.
(b) $\rho(x,y)=\rho(y,x)$
(c) $\forall x,y,z \in M$, $\rho(x,y) \leq \rho(x,z)+\rho(y,z)$.

Proof of exercise 5.6.6

(a) Take $x,y \in (0,\infty) \ni x=y \Rightarrow \rho(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{x}-\frac{1}{x}\right| = |0| = 0$

Take $x,y \in (0,\infty) \ni x \neq y \Rightarrow \rho(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| > 0$ since the numerator is clearly not zero and since $\frac{y-x}{xy} \in \mathbf{R}$ and by the properties of $|\cdot|$.

(b) $\rho(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right| = \left|-\left(\frac{1}{y}-\frac{1}{x}\right)\right| = |-1|\left|\frac{1}{y}-\frac{1}{x}\right| = \left|\frac{1}{y}-\frac{1}{x}\right| = \rho(y,x)$

(c) Take $x,y,z \in (0,\infty)$

$\rho(x,y)=\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{x}-\frac{1}{z}+\frac{1}{z}-\frac{1}{y}\right| \leq \left|\frac{1}{x}-\frac{1}{z}\right| + \left|\frac{1}{z}-\frac{1}{y}\right|$ (Triangle inequality of $\mathbf{R}$)

$= \left|\frac{1}{x}-\frac{1}{z}\right| + \left|\frac{1}{y}-\frac{1}{z}\right| = \rho(x,y)+\rho(y,z)$. Q.E.D.