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 Problem Set 4

5.6.5 Suppose that  $\rho_1$  and  $\rho_2$  are metrics on  $M$ . Prove that the following are also metrics:

A metric satisfies the following three properties:

(a)  $\forall x, y \in M, \rho(x, y) \geq 0$  and  $\rho(x, y) = 0$  if and only if  $x = y$ .

(b)  $\rho(x, y) = \rho(y, x)$

(c)  $\forall x, y, z \in M, \rho(x, y) \leq \rho(x, z) + \rho(y, z)$ .

5.6.5.a  $\rho \equiv \rho_1 + \rho_2$

Proof of exercise 5.6.5.a:

Since  $\rho_1$  and  $\rho_2$  are metrics, they satisfy the three properties listed above.

(a) Take  $x, y \in M \ni x \neq y$ .  $\rho_1(x, y) > 0$  and  $\rho_2(x, y) > 0$ .

$\Rightarrow \rho(x, y) = \rho_1(x, y) + \rho_2(x, y) > 0$

Take  $x = y$ .  $\rho_1(x, y) = 0$  and  $\rho_2(x, y) = 0$

$\Rightarrow \rho(x, y) = \rho_1(x, y) + \rho_2(x, y) = 0 + 0 = 0$

(b)  $\rho(x, y) = \rho_1(x, y) + \rho_2(x, y) = \rho_1(y, x) + \rho_2(y, x) = \rho(y, x)$

(c) Take  $x, y, z \in M$ .

$\rho_1(x, y) \leq \rho_1(x, z) + \rho_1(y, z); \rho_2(x, y) \leq \rho_2(x, z) + \rho_2(y, z)$

$\Rightarrow \rho(x, y) = \rho_1(x, y) + \rho_2(x, y) \leq \rho_1(x, z) + \rho_1(y, z) + \rho_2(x, z) + \rho_2(y, z)$

$= \rho_1(x, z) + \rho_2(x, z) + \rho_1(y, z) + \rho_2(y, z) = \rho(x, z) + \rho(y, z)$ . Q.E.D.

5.6.5.b  $\rho' \equiv \max\{\rho_1, \rho_2\}$

Since  $\rho_1$  and  $\rho_2$  are metrics, they satisfy the three properties listed above.

(a) Take  $x, y \in M \ni x \neq y$ .  $\rho_1(x, y) > 0$  and  $\rho_2(x, y) > 0$ .

$\Rightarrow \rho(x, y) = \max\{\rho_1(x, y), \rho_2(x, y)\} > 0$

Take  $x = y$ .  $\rho_1(x, y) = 0$  and  $\rho_2(x, y) = 0$

$\Rightarrow \rho(x, y) = \max\{\rho_1(x, y), \rho_2(x, y)\} = \max\{0, 0\} = 0$ .

(b)  $\rho(x, y) = \max\{\rho_1(x, y), \rho_2(x, y)\} = \max\{\rho_1(y, x), \rho_2(y, x)\} = \rho(y, x)$

(c) Take  $x, y, z \in M$ .

$\rho_1(x, y) \leq \rho_1(x, z) + \rho_1(y, z); \rho_2(x, y) \leq \rho_2(x, z) + \rho_2(y, z)$

$\Rightarrow \rho(x, y) = \max\{\rho_1(x, y), \rho_2(x, y)\} \leq \max\{\rho_1(x, z) + \rho_1(y, z), \rho_2(x, z) + \rho_2(y, z)\}$

$\leq \max\{\rho_1(x, z), \rho_2(x, z)\} + \max\{\rho_1(y, z), \rho_2(y, z)\} = \rho(x, z) + \rho(y, z)$ . Q.E.D.