5.6.12 Prove that the metrics ρ_1, $\rho_{\max}$, and ρ_2 defined in Example 3 are uniformly equivalent.

Recall: $\rho_1((x_1,y_1),(x_2,y_2)) \equiv |x_1 - x_2| + |y_1 - y_2|$

$\rho_2((x_1,y_1),(x_2,y_2)) \equiv \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$

$\rho_{\max}((x_1,y_1),(x_2,y_2)) \equiv \max\{|x_1 - x_2|,|y_1 - y_2|\}$

Lemma 5.6.12:

Uniform equivalence is an equivalence relation.

Proof of lemma 5.6.12:

Take ρ_1, ρ_2, ρ_2 metrics on M.

i) ρ_1 is uniformly equivalent to ρ_1:

Let $c_1 = c_2 = 1 \Rightarrow 1\rho_1(x,y) \le \rho_1(x,y) \le 1\rho_1(x,y) \Leftrightarrow \rho_1(x,y) = \rho_1(x,y)$, which is always true.

ii) If ρ_1 is uniformly equivalent to ρ_2, then ρ_2 is uniformly equivalent to ρ_2 :

ρ_1 uniformly equivalent to $\rho_2 \Rightarrow \exists\, c_1, c_2 > 0 \ni c_1\rho_1(x,y) \le \rho_2(x,y) \le c_2\rho_1(x,y)$ $\forall x, y \in M$.

Take $c_3 = \dfrac{1}{c_2}$ and $c_4 = \dfrac{1}{c_1}$.

Then $c_3\rho_2(x,y) = \dfrac{1}{c_2}\rho_2(x,y) \le \dfrac{c_2}{c_2}\rho_1(x,y) = \rho_1(x,y)$ $\forall x, y \in M$ and

$c_4\rho_2(x,y) = \dfrac{1}{c_1}\rho_2(x,y) \ge \dfrac{c_1}{c_1}\rho_1(x,y) = \rho_1(x,y)$ $\forall x, y \in M$

$\Rightarrow c_3\rho_2(x,y) \le \rho_1(x,y) \le c_4\rho_2(x,y)$ $\forall x, y \in M$. i.e ρ_2 is uniformly equivalent to ρ_1.

iii) If ρ_1 is uniformly equivalent to ρ_2 and ρ_2 is uniformly equivalent to ρ_3, then ρ_1 is uniformly equivalent to ρ_3.

ρ_1 uniformly equivalent to $\rho_2 \Rightarrow \exists\, c_1, c_2 > 0 \ni c_1\rho_1(x,y) \le \rho_2(x,y) \le c_2\rho_1(x,y)$ $\forall x, y \in M$.

ρ_2 uniformly equivalent to $\rho_3 \Rightarrow \exists\, c_3, c_4 > 0 \ni c_3\rho_2(x,y) \le \rho_3(x,y) \le c_4\rho_2(x,y)$ $\forall x, y \in M$.

Take $c_5 = c_1c_3 \Rightarrow c_5\rho_1(x,y) = c_1c_3\rho_1(x,y) \le c_3\rho_2(x,y) \le \rho_3(x,y)$ $\forall x, y \in M$.

Take $c_6 = c_2c_4 \Rightarrow c_6\rho_1(x,y) = c_2c_4\rho_1(x,y) \ge c_4\rho_2(x,y) \ge \rho_3(x,y)$ $\forall x, y \in M$.

$\Rightarrow c_5\rho_1(x,y) \le \rho_3(x,y) \le c_6\rho_1(x,y)$ $\forall x, y \in M$. That is, ρ_1 is uniformly equivalent to ρ_3. Q.E.D.

Proof of exercise 5.6.12: ρ_2 is uniformly equivalent to ρ_1.

Take $x, y \in M$ arbitrary. Then we have:

$(\rho_2(x,y))^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2 \le |x_1 - x_2|^2 + 2|x_1 - x_2||y_1 - y_2| + |y_1 - y_2|^2$

$= (|x_1 - x_2| + |y_1 - y_2|)^2 = (\rho_1(x,y))^2$

$\Rightarrow \rho_2(x,y) \le \rho_1(x,y)$. Thus, we can take $c_1 = 1$.

Next, we have:

$(\sqrt{2}\rho_2(x,y))^2 = 2|x_1 - x_2|^2 + 2|y_1 - y_2|^2$

$= |x_1 - x_2|^2 + |x_1 - x_2|^2 + |y_1 - y_2|^2 + |y_1 - y_2|^2$

$\ge |x_1 - x_2|^2 + 2|x_1 - x_2||y_1 - y_2| + |y_1 - y_2|^2$ (by exercise 1.1.8)

$= (|x_1 - x_2| + |y_1 - y_2|)^2 = (\rho_1(x,y))^2$

$\Rightarrow \sqrt{2}\rho_2(x,y) \ge \rho_1(x,y)$. Thus, we can take $c_2 = \sqrt{2}$.

$\Rightarrow \rho_2(x,y) \le \rho_1(x,y) \le \sqrt{2}\rho_2(x,y)$. Since $x, y \in M$ was arbitrary, this holds for all x, y and we have that ρ_2 is uniformly equivalent to ρ_1.

Proof of exercise 5.6.12: $\rho_{\max}$ is uniformly equivalent to ρ_2.

Take $x, y \in M$ arbitrary. Then we have:

$\rho_2(x,y) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} \ge \sqrt{|x_1 - x_2|^2} = |x_1 - x_2|$ and

$\rho_2(x,y) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} \ge \sqrt{|y_1 - y_2|^2} = |y_1 - y_2|$

$\Rightarrow \rho_2(x,y) \ge \max\{|x_1 - x_2|, |y_1 - y_2|\} = \rho_{\max}(x,y)$. Thus, we can take $c_1 = 1$.

Next, we have:

Without loss of generality, assume $|x_1 - x_2| \ge |y_1 - y_2|$. (An identical argument holds for the case where $|y_1 - y_2| \ge |x_1 - x_2|$.

$\Rightarrow (\sqrt{2}\rho_{\max}(x,y))^2 = (\max\{\sqrt{2}|x_1 - x_2|, \sqrt{2}|y_1 - y_2|\})^2$

$\ge 2|x_1 - x_2|^2 = |x_1 - x_2|^2 + |x_1 - x_2|^2 \ge |x_1 - x_2|^2 + |y_1 - y_2|^2 = (\rho_2(x,y))^2$

$\Rightarrow \sqrt{2}\rho_{\max}(x,y) \ge \rho_2(x,y)$. Thus, we can take $c_2 = \sqrt{2}$.

$\Rightarrow \rho_{\max}(x,y) \le \rho_2(x,y) \le \sqrt{2}\rho_{\max}(x,y)$. Since $x, y \in M$ was arbitrary, this holds for all x, y and we have that $\rho_{\max}$ is uniformly equivalent to ρ_2.

Proof of exercise 5.6.12: ρ_1 is uniformly equivalent to $\rho_{\max}$.

Take $x, y \in M$ arbitrary. Then we have:

$\rho_1(x,y) = |x_1 - x_2| + |y_1 - y_2| \ge |x_1 - x_2|$ and

$\rho_1(x,y) = |x_1 - x_2| + |y_1 - y_2| \ge |y_1 - y_2|$

$\Rightarrow \rho_1(x,y) \ge \max\{|x_1 - x_2|, |y_1 - y_2|\} = \rho_{\max}(x,y)$. Thus, we can take $c_1 = 1$.

Without loss of generality, assume $|x_1 - x_2| \ge |y_1 - y_2|$. (An identical argument holds for the case where $|y_1 - y_2| \ge |x_1 - x_2|$.

$\Rightarrow 2\rho_{\max}(x,y) = \max\{2|x_1 - x_2|, 2|y_1 - y_2|\} \ge 2|x_1 - x_2| \ge |x_1 - x_2| + |y_1 - y_2|$

$= \rho_1(x, y)$. Thus, we can take $c_2 = 2$.
$\Rightarrow \rho_{\max}(x, y) \le \rho_1(x, y) \le 2\rho_{\max}(x, y)$. Since $x, y \in M$ was arbitrary, this holds for all x, y and we have that $\rho_{\max}$ is uniformly equivalent to ρ_1.

Conclusion of proof:

By lemma 5.6.12, uniform equivalence is an equivalence relation. Thus, we have:
ρ_1 is uniformly equivalent to ρ_2
ρ_2 is uniformly equivalent to ρ_1
ρ_1 is uniformly equivalent to $\rho_{\max}$
$\rho_{\max}$ is uniformly equivalent to ρ_1
ρ_2 is uniformly equivalent to $\rho_{\max}$
$\rho_{\max}$ is uniformly equivalent to ρ_2. Q.E.D.