5.6.11 Two metrics, ρ and σ, on a set *M* are said to be **uniformly equivalent** if there exist positive constants c_1 and c_2 such that

$$c_1\rho(x,y) \le \sigma(x,y) \le c_2\rho(x,y)$$

for all *x* and *y* in *M*. Prove that if ρ and σ are uniformly equivalent, then they are equivalent.

Lemma 5.6.11.a:

If ρ is uniformly equivalent to σ, then σ is uniformly equivalent to ρ.

Proof of lemma 5.6.11.a

ρ uniformly equivalent to $\sigma \Rightarrow \exists\, c_1, c_2 > 0 \ni c_1\rho(x,y) \le \sigma(x,y) \le c_2\rho(x,y)$ $\forall x, y \in M$.

Take $c_3 = \dfrac{1}{c_2}$ and $c_4 = \dfrac{1}{c_1}$.

Then $c_3\sigma(x,y) = \dfrac{1}{c_2}\sigma(x,y) \le \dfrac{c_2}{c_2}\rho(x,y) = \rho(x,y)$ $\forall x, y \in M$ and

$c_4\sigma(x,y) = \dfrac{1}{c_1}\sigma(x,y) \ge \dfrac{c_1}{c_1}\rho(x,y) = \rho(x,y)$ $\forall x, y \in M$

$\Rightarrow c_3\sigma(x,y) \le \rho(x,y) \le c_4\sigma(x,y)$ $\forall x, y \in M$. i.e σ is uniformly equivalent to ρ. Q.E.D.

Proof of exercise 5.6.11

Let ρ and σ be metrics on a set *M* which are uniformly equivalent. That is, $\exists\, c_1, c_2 > 0 \ni c_1\rho(x,y) \le \sigma(x,y) \le c_2\rho(x,y)$. In addition, by lemma 5.6.11.a, we have that $\dfrac{1}{c_2}\sigma(x,y) \le \rho(x,y) \le \dfrac{1}{c_1}\sigma(x,y)$.

Take $\varepsilon > 0$ arbitrary and $x \in M$ arbitrary.

Let $\delta = \min\left\{\varepsilon\, c_1, \dfrac{\varepsilon}{c_2}\right\}$.

Suppose $\rho(x,y) \le \delta \le \dfrac{\varepsilon}{c_2} \Rightarrow \sigma(x,y) \le c_2\rho(x,y) \le c_2\dfrac{\varepsilon}{c_2} = \varepsilon$.

Suppose $\sigma(x,y) \le \delta \le \varepsilon\, c_1 \Rightarrow \rho(x,y) \le \dfrac{1}{c_1}\sigma(x,y) \le \dfrac{1}{c_1}\varepsilon\, c_1 = \varepsilon$.

i.e. ρ and σ are equivalent. Q.E.D.