5.3.9 Let Q be a closed, finite, rectangle in the plane, and let $C(Q)$ denote the set of real-valued continuous functions on Q. For f in $C(Q)$, define

$$\|f\|_\infty = \sup_{(x,y)\in Q} |f(x,y)|.$$

5.3.9.a Prove that this norm satisfies the properties (a), (b), and (c) of proposition 5.3.1.

5.3.9.a.a Prove that $\|f\|_\infty \geq 0$ and $\|f\|_\infty = 0$ if and only if f is the zero function on Q.

Proof of exercise 5.3.9.a.a:

$\|f\|_\infty = \sup_{(x,y)\in Q} |f(x,y)|$. Since $|f(x,y)| \geq 0\ \forall (x,y) \in Q$, we have: $\sup_{(x,y)\in Q} |f(x,y)| \geq 0$.

Clearly, $\sup_{(x,y)\in Q} |f(x,y)| = 0 \Leftrightarrow f(x,y) = 0\ \forall (x,y) \in Q$. Thus, $\|f\|_\infty = 0$ if and only if f is identically zero. Q.E.D.

5.3.9.a.b For every $\alpha \in \mathbf{R}$, we have $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty$.

Proof of exercise 5.3.9.a.b:

$\|\alpha f\|_\infty = \sup_{(x,y)\in Q} |\alpha f(x,y)| = |\alpha| \sup_{(x,y)\in Q} |f(x,y)| = |\alpha| \|f\|_\infty$ by exercise 2.5.8. Q.E.D.

5.3.9.a.c $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Proof of exercise 5.3.9.a.c:

$\forall (x,y) \in Q$, we have $|f(x,y) + g(x,y)| \leq |f(x,y)| + |g(x,y)|$. In particular, we have that $\sup_{(x,y)\in Q} |f(x,y) + g(x,y)| \leq |f(x,y)| + |g(x,y)|$.

In addition, $\forall (x,y) \in Q$, $|f(x,y)| \leq \sup_{(x,y)\in Q} |f(x,y)|$, $|g(x,y)| \leq \sup_{(x,y)\in Q} |g(x,y)|$.

Thus, $\sup_{(x,y)\in Q} |f(x,y) + g(x,y)| \leq |f(x,y)| + |g(x,y)| \leq \sup_{(x,y)\in Q} |f(x,y)| + \sup_{(x,y)\in Q} |g(x,y)|$.

i.e. $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Q.E.D.

5.3.9.b Prove that $C(Q)$ is complete.

Proof of exercise 5.3.9.b:

$C(Q)$ is complete if every Cauchy sequence of elements of $C(Q)$ converges to an element of $C(Q)$.

Let $\{f_n\}$ be a Cauchy sequence of elements of $C(Q)$.

That is, $\forall \varepsilon > 0, \exists N(\varepsilon) > 0 \ni \forall n \geq N, \|f_n - f_m\|_\infty < \varepsilon$.

By definition,

$\forall (x,y) \in Q, |f_n(x,y) - f_m(x,y)| \leq \sup_{(x,y)\in Q} |f_n(x,y) - f_m(x,y)| = \|f_n - f_m\|_\infty$.

Thus, $\forall n \geq N$, $|f_n(x,y) - f_m(x,y)| < \varepsilon$. That is, for any fixed (x,y), the sequence $\{f_n(x,y)\}$ is a Cauchy sequence of real numbers, which by the completeness of the real numbers, converges to some real number $f(x,y)$.

Since the absolute value is a continuous function, it follows that:
$|f(x,y) - f_m(x,y)| = \lim_{n\to\infty} |f_n(x,y) - f_m(x,y)|$. Thus,
$|f(x,y) - f_m(x,y)| < \varepsilon \ \forall m \geq N \ \forall (x,y) \in Q$. Therefore,
$\|f - f_m\|_\infty < \varepsilon \ \forall m \geq N \ \forall (x,y) \in Q$. i.e f_m converges to f in the sup norm.
Therefore, f_m converges to f uniformly on Q by proposition 5.3.2 $\Rightarrow f$ is continuous by theorem 5.2.1. That is, every Cauchy sequence in $C(Q)$ converges to a function in $C(Q)$. i.e. $C(Q)$ equipped with $\|\cdot\|_\infty$ is complete. Q.E.D.